4 # Trigonometria

En context (pàg. 99)

a) Tales es va valer únicament d'un bastó, d'una corda i d'un ajudant. Va calcular que l'ombra projectada per la seva alçada tindria una proporció similar a l'ombra de la mateixa piràmide respecte a l'alçària d'aquesta. Es pot ampliar aquesta història en la pàgina web següent: **http://links.edebe.com/2buc**

Fixa-t'hi (pàg. 102)

— $\frac{\overline{OA}}{\overline{OP}} = \frac{\overline{OA'}}{\overline{OP'}} = \cos\alpha$

— $\frac{\overline{AP}}{\overline{OA}} = \frac{\overline{A'P'}}{\overline{OA'}} = \operatorname{tg}\alpha$

Problemes resolts (pàgs. 114 i 115)

1. Per a calcular les raons trigonomètriques de l'angle de 60° a partir de l'angle de 30°, utilitzem les raons trigonomètriques d'angles complementaris. Sigui $\alpha = 60°$ i $\beta = 30°$, aleshores, $\alpha + \beta = 90°$. Ara, calculem les raons:

$$\sin\alpha = \cos\beta \rightarrow \sin 60° = \cos 30° = \frac{\sqrt{3}}{2} \rightarrow \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos\alpha = \sin\beta \rightarrow \cos 60° = \sin 30° = \frac{1}{2} \rightarrow \cos 60° = \frac{1}{2}$$

$$\operatorname{tg}\alpha = \frac{1}{\operatorname{tg}\beta} \rightarrow \operatorname{tg} 60° = \frac{1}{\operatorname{tg} 30°} = \frac{1}{\frac{\sqrt{3}}{3}} = \sqrt{3} \rightarrow \operatorname{tg} 60° = \sqrt{3}$$

2. Hem de tenir en compte que, com que α és un angle del quart quadrant, els valors del sinus i la tangent seran negatius.

Teorema fonamental de la trigonometria:

$$\sin^2\alpha + \cos^2\alpha = 1 \Rightarrow \sin^2\alpha + \left(\frac{1}{4}\right)^2 = 1$$

$$\Rightarrow \sin\alpha = -\sqrt{1 - \frac{1}{16}} = -\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$$

Calculem la resta de raons trigonomètriques:

$$\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{-\frac{\sqrt{15}}{4}}{\frac{1}{4}} = -\sqrt{15} \Rightarrow$$

$$\Rightarrow \operatorname{cotg}\alpha = \frac{1}{\operatorname{tg}\alpha} = -\frac{\sqrt{15}}{15}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{\frac{1}{4}} = 4 \Rightarrow$$

$$\Rightarrow \operatorname{cosec}\alpha = \frac{1}{\sin\alpha} = -\frac{1}{\frac{\sqrt{15}}{4}} = -\frac{4\sqrt{15}}{15}$$

3. Expressem 3α com la suma dels angles α i 2α, i hi apliquem la fórmula del cosinus de la suma de dos angles i la fórmula del sinus i el cosinus de l'angle doble.

$$\cos 3\alpha = \cos(\alpha + 2\alpha) = \cos\alpha \cdot \cos 2\alpha - \sin\alpha \cdot \sin 2\alpha =$$
$$= \cos\alpha \cdot (\cos^2\alpha - \sin^2\alpha) - \sin\alpha \cdot (2\sin\alpha\cos\alpha) =$$
$$= \cos^3\alpha - \cos\alpha \cdot \sin^2\alpha - 2\sin^2\alpha\cos\alpha =$$
$$= \cos^3\alpha - 3\sin^2\alpha\cos\alpha$$

Per tant, $\cos 3\alpha = \cos^3\alpha - 3\sin^2\alpha\cos\alpha$.

4. Per a resoldre aquest exercici, necessitem calcular el $\sin\alpha$ i després utilitzar les fórmules de l'exercici 3 i de l'exercici B d'aquesta pàgina.

Sabem que $90° < \alpha < 180°$, per tant, l'angle està situat en el segon quadrant, la qual cosa significa que $\sin\alpha > 0$.

Apliquem el teorema fonamental de la trigonometria per a calcular el $\sin\alpha$:

$$\sin^2\alpha + \cos^2\alpha = 1 \Rightarrow \sin^2\alpha + (-0{,}6)^2 = 1$$
$$\Rightarrow \sin\alpha = \sqrt{1 - 0{,}36} = 0{,}8$$

Calculem $\sin 3\alpha$, $\cos 3\alpha$ i $\operatorname{tg} 3\alpha$:

$$\sin 3\alpha = 3\sin\alpha\cos^2\alpha - \sin^3\alpha = 3 \cdot 0{,}8 \cdot (-0{,}6)^2 - 0{,}8^3 = 0{,}35$$
$$\cos 3\alpha = \cos^3\alpha - 3\sin^2\alpha\cos\alpha = (-0{,}6)^3 - 3 \cdot 0{,}8^2(-0{,}6) = 0{,}94$$
$$\operatorname{tg} 3\alpha = \frac{\sin 3\alpha}{\cos 3\alpha} = \frac{0{,}35}{0{,}94} = 0{,}37$$

5. a) $\sin x \cdot (\operatorname{cosec} x - \sin x) = \cos^2 x$

$$\operatorname{cosec} x = \frac{1}{\sin x} \Rightarrow \sin x \cdot (\operatorname{cosec} x - \sin x) =$$
$$= \sin x \cdot \left(\frac{1}{\sin x} - \sin x\right) = 1 - \sin^2 x = \cos^2 x$$

b) $\frac{\cos x}{1 - \sin x} - \operatorname{tg} x = \sec x$

$$\operatorname{tg} x = \frac{\sin x}{\cos x} \Rightarrow \frac{\cos x}{1 - \sin x} - \operatorname{tg} x = \frac{\cos x}{1 - \sin x} - \frac{\sin x}{\cos x} =$$
$$= \frac{\cos^2 x - (1 - \sin x) \cdot \sin x}{(1 - \sin x) \cdot \cos x} = \frac{\cos^2 x + \sin^2 x - \sin x}{(1 - \sin x) \cdot \cos x} =$$
$$= \frac{\cancel{(1 - \sin x)}}{\cancel{(1 - \sin x)} \cdot \cos x} = \frac{1}{\cos x} = \sec x$$

c) $\frac{1}{\sec^2\alpha} = \sin^2\alpha\cos^2\alpha + \cos^4\alpha$

$$\sec\alpha = \frac{1}{\cos\alpha} \Rightarrow \frac{1}{\sec^2\alpha} = \frac{1}{\frac{1}{\cos^2\alpha}} = \cos^2\alpha = 1 \cdot \cos^2\alpha =$$
$$= (\cos^2\alpha + \sin^2\alpha) \cdot \cos^2\alpha = \cos^4\alpha + \sin^2\alpha\cos^2\alpha$$

d) $\frac{\sin\alpha}{\operatorname{cosec}\alpha} + \frac{\cos\alpha}{\sec\alpha} = 1$

$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha},\ \sec\alpha = \frac{1}{\cos\alpha} \Rightarrow \frac{\sin\alpha}{\operatorname{cosec}\alpha} + \frac{\cos\alpha}{\sec\alpha} =$$

$$= \frac{\sin\alpha}{\frac{1}{\sin\alpha}} + \frac{\cos\alpha}{\frac{1}{\cos\alpha}} = \sin^2\alpha + \cos^2\alpha = 1$$

6. $\sin x + \sqrt{3}\cos x = 2 \Rightarrow \frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = 1$

$$\sin(x + 60°) = 1 \Rightarrow x + 60° = 90° + 360° \cdot k \Rightarrow$$
$$\Rightarrow x = 30° + 360° \cdot k$$

7. $\left.\begin{array}{l} \sin x \cdot \cos y = \frac{3}{4} \\ \cos x \cdot \sin y = \frac{1}{4} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{1}{2}(\sin(x+y) + \sin(x-y)) = \frac{3}{4} \\ \frac{1}{2}(\sin(x+y) - \sin(x-y)) = \frac{1}{4} \end{array}\right\}$

$$\Rightarrow \left.\begin{array}{l} \sin(x+y) + \sin(x-y) = \frac{3}{2} \\ \sin(x+y) - \sin(x-y) = \frac{1}{2} \end{array}\right\} \Rightarrow 2\sin(x+y) = 2$$

$$\Rightarrow \sin(x+y) = 1 \Rightarrow x + y = 90° + 360° \cdot k$$

$$\left.\begin{array}{l} \sin(x+y) + \sin(x-y) = \frac{3}{2} \\ \sin(x+y) - \sin(x-y) = \frac{1}{2} \end{array}\right\} \Rightarrow 2\sin(x-y) = 1$$

$$\Rightarrow \sin(x-y) = \frac{1}{2} \Rightarrow x - y = 30° + 360° \cdot k$$

$$\Rightarrow \begin{cases} x + y = 90° + 360° \cdot k \\ x - y = 30° + 360° \cdot k \end{cases} \Rightarrow \begin{cases} x = 60° + 180° \cdot k \\ y = 30° + 180° \cdot k \end{cases}$$

Exercicis i problemes (pàgs. 116 a 120)

1 ANGLES I MESURES — Pàg. 116

8. a) $315° = 315\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{7}{4}\pi \text{ rad} = 5{,}5 \text{ rad}$

b) $300° = 300\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{5}{3}\pi \text{ rad} = 5{,}24 \text{ rad}$

c) $2\,210° = 2\,210\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{221}{18}\pi \text{ rad} = 38{,}57 \text{ rad}$

d) $1\,650° = 1\,650\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{55}{6}\pi \text{ rad} = 28{,}8 \text{ rad}$

9. a) $\frac{\pi}{6} \text{ rad} = \frac{\pi}{6} \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 30°$

b) $\frac{9\pi}{5} \text{ rad} = \frac{9\pi}{5} \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 324°$

c) $\frac{17\pi}{18} \text{ rad} = \frac{17\pi}{18} \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 170°$

d) $\frac{215\pi}{4} \text{ rad} = \frac{215\pi}{4} \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 9\,675°$

10. $134° = 134\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{67}{90}\pi \text{ rad} = 2{,}34 \text{ rad}$

$4{,}3\pi \text{ rad} = 4{,}3\pi \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 774°$

$350° = 350\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{35}{18}\pi \text{ rad} = 6{,}11 \text{ rad}$

$0{,}5\pi \text{ rad} = 0{,}5\pi \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 90°$

11. Per a reduir a primer gir hem de restar 360° a l'angle, si està expressat en graus, i 2π rad, si l'angle està expressat en radians. Es duu a terme aquest procediment fins que el grau es trobi entre 0° i 360°, o entre 0 i 2π rad, respectivament.

a) $529° - 360° = 169°$

b) La part entera de (2 952° : 360°) és 8; per tant, efectuem:

$$2\,952° - (8 \cdot 360°) = 72°$$

c) La part entera de $\left(\frac{55}{6} : 2\right)$ és 4; per tant, efectuem:

$$\frac{55\pi}{6} \text{ rad} - (4 \cdot 2\pi \text{ rad}) = \frac{7\pi}{6} \text{ rad}$$

d) La part entera de $\left(\frac{217}{4} : 2\right)$ és 27; per tant, efectuem:

$$\frac{217\pi}{4} \text{ rad} - (27 \cdot 2\pi \text{ rad}) = \frac{\pi}{4} \text{ rad}$$

12. $\frac{7\pi}{5} \text{ rad} = \frac{7\pi}{5} \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 252°$

$160° = 160\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{8\pi}{9} \text{ rad}$

13. Hem de convertir els minuts i els segons en graus i sumar-los.

a) $\alpha = 25°\ 9'\ 53''$

$$9' = 9' \cdot \frac{1°}{60'} = 0{,}15° \qquad 53'' = 53'' \cdot \frac{1°}{3\,600''} = 0{,}014\,7\widehat{2}°$$
$$\alpha = 25° + 0{,}15° + 0{,}014\,7\widehat{2}° = 25{,}164\,7\widehat{2}$$

b) $\beta = 46°\ 31'\ 18''$

$$31' = 31' \cdot \frac{1°}{60'} = 0{,}51\widehat{6}° \qquad 18'' = 18'' \cdot \frac{1°}{3\,600''} = 0{,}005°$$
$$\beta = 46° + 0{,}51\widehat{6}° + 0{,}005° = 46{,}521\widehat{6}°$$

14. Hem de convertir la part decimal en minuts i segons.

a) $\alpha = 15{,}722\,5°$

$$\left.\begin{array}{l} 0{,}722\,5° = 0{,}722\,5° \cdot \frac{60'}{1°} = 43{,}35' \\ 0{,}35' = 0{,}35' \cdot \frac{60''}{1'} = 21'' \end{array}\right\} \rightarrow \alpha = 15°\ 43'\ 21''$$

b) $\beta = 36{,}5736°$

$$\left.\begin{array}{l} 0{,}5736° = 0{,}5736° \cdot \dfrac{60'}{1°} = 34{,}416' \\ 0{,}416' = 0{,}416' \cdot \dfrac{60''}{1'} = 24{,}96'' \end{array}\right\} \rightarrow \beta = 36°\ 34'\ 24{,}96''$$

15. a) $2\alpha + \beta = 2 \cdot (20°\ 12'\ 30'') + (10°\ 20'\ 15'') = (40°\ 24'\ 60'') +$
$+ (10°\ 20'\ 15'') = (40°\ 25') + (10°\ 20'\ 15'') = 50°\ 45'\ 15''$

b) Hem de convertir els graus i els minuts en segons.

$$50° = 50° \cdot \frac{3600''}{1°} = 180\,000''\quad 45' = 45' \cdot \frac{60''}{1'} = 2700''$$
$$2\alpha + \beta = 180\,000'' + 2700'' + 15'' = 182\,715''$$

2 RAONS TRIGONOMÈTRIQUES D'UN ANGLE AGUT

Pàgs. 116 i 117

16. a) $\sin \alpha = 0{,}608 \rightarrow \alpha = \text{arc sin } 0{,}608 = 37{,}445°$

b) $\cos \alpha = 0{,}978 \rightarrow \alpha = \text{arc cos } 0{,}978 = 12{,}041°$

c) $\text{tg } \alpha = 2{,}365 \rightarrow \alpha = \text{arc tg } 2{,}365 = 67{,}08°$

d) $\sin \alpha = 0{,}123 \rightarrow \alpha = \text{arc sin } 0{,}123 = 7{,}065°$

17. Sabem que la raó que relaciona el catet oposat amb la hipotenusa és el sinus; per tant:

$$\sin \alpha = \frac{1}{2{,}5} = 0{,}4 \Rightarrow \alpha = \text{arc } \sin(0{,}4) = 23{,}578°$$

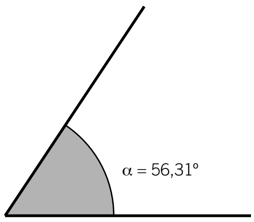
18. $\text{cosec } \alpha = \dfrac{1}{\sin \alpha} \Rightarrow \text{cosec } 45° = \dfrac{1}{\sin 45°} = \dfrac{1}{\frac{\sqrt{2}}{2}} =$

$= \dfrac{2}{\sqrt{2}} = \sqrt{2}$

19. L'únic catet el valor del qual coneixem és el catet oposat a l'angle de 31°, i el catet oposat es relaciona amb la hipotenusa mitjançant el sinus. Per tant, si *h* és la hipotenusa:

$$\sin 31° = \frac{3}{h} \Rightarrow h = \frac{3}{\sin 31°} \Rightarrow h = 5{,}82$$

20. $\text{tg } \alpha = \dfrac{3}{2} \Rightarrow \alpha = \text{arc tg}\left(\dfrac{3}{2}\right) \Rightarrow \alpha = 56{,}31°$



21. $\sin \alpha = \dfrac{\text{c. oposat}}{\text{hipotenusa}} = \dfrac{4}{5} \Rightarrow \text{cosec } \alpha = \dfrac{1}{\sin \alpha} = \dfrac{1}{\frac{4}{5}} = \dfrac{5}{4}$

$$\cos \alpha = \frac{\text{c. contigu}}{\text{hipotenusa}} = \frac{3}{5} \Rightarrow \sec \alpha = \frac{1}{\cos \alpha} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$$

$$\text{tg } \alpha = \frac{\text{c. oposat}}{\text{c. contigu}} = \frac{4}{3} \Rightarrow \text{cotg } \alpha = \frac{1}{\text{tg } \alpha} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

22. Per a calcular les raons trigonomètriques de l'angle de 60° a partir de l'angle de 30°, utilitzem les raons trigonomètriques d'angles complementaris. Sigui $\alpha = 60°$ i $\beta = 30°$, aleshores, $\alpha + \beta = 90°$. Ara, calculem les raons:

$$\sin \alpha = \cos \beta \rightarrow \sin 60° = \cos 30° = \frac{\sqrt{3}}{2} \rightarrow \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos \alpha = \sin \beta \rightarrow \cos 60° = \sin 30° = \frac{1}{2} \rightarrow \cos 60° = \frac{1}{2}$$

$$\text{tg } \alpha = \frac{1}{\text{tg } \beta} \rightarrow \text{tg } 60° = \frac{1}{\text{tg } 30°} = \frac{1}{\frac{\sqrt{3}}{3}} = \sqrt{3} \rightarrow \text{tg } 60° = \sqrt{3}$$

23. $\cos \alpha = 0{,}6588 \Rightarrow \alpha = \text{arc } \cos(0{,}6588) = 48{,}79°$

Apliquem el teorema fonamental de la trigonometria per a calcular el sin α:

$$\sin^2 \alpha + \cos^2 \alpha = 1 \Rightarrow \sin^2 \alpha + (0{,}6588)^2 = 1 \Rightarrow$$
$$\Rightarrow \sin \alpha = 0{,}7523$$
$$\Rightarrow \text{tg } \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{0{,}7523}{0{,}6588} = 1{,}1419$$

3 RAONS TRIGONOMÈTRIQUES D'UN ANGLE QUALSEVOL

Pàg. 117

24. Calculem les raons trigonomètriques restants de l'angle de 23°.

$$\sin^2 23° + \cos^2 23° = 1 \Rightarrow (0{,}39)^2 + \cos^2 23° = 1$$
$$\Rightarrow \cos 23° = \sqrt{1 - (0{,}39)^2} = 0{,}92$$
$$\text{tg } 23° = \frac{\sin 23°}{\cos 23°} = \frac{0{,}39}{0{,}92} = 0{,}42$$

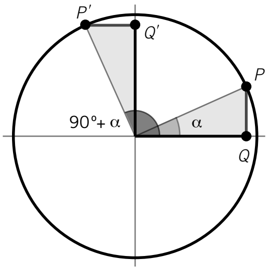
a) Els angles de 23° i 67° són complementaris; per tant:

$$\cos 67° = \sin 23° = 0{,}39$$

b) Per la mateixa raó que en l'apartat (a):

$$\text{tg } 67° = \frac{1}{\text{tg } 23°} = \frac{1}{0{,}42} = 2{,}38$$

25.



A partir d'aquesta figura, podem veure clarament que les raons trigonomètriques de l'angle $\frac{\pi}{2}+\alpha$ són:

$$\sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha;\ \cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha;$$

$$\operatorname{tg}\left(\frac{\pi}{2}+\alpha\right)=\frac{-1}{\operatorname{tg}\alpha}$$

26. Calculem les raons trigonomètriques de l'angle de 75° aplicant la fórmula de l'angle suma:

$$\sin 75° = \sin(45°+30°) = \sin 45°\cos 30° + \cos 45°\sin 30° =$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\cos 75° = \cos(45°+30°) = \cos 45°\cos 30° - \sin 45°\sin 30° =$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\operatorname{tg} 75° = \frac{\sin(45°+30°)}{\cos(45°+30°)}=\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}=\frac{\left(\sqrt{6}+\sqrt{2}\right)^2}{4}$$

Com que els angles de 75° i de 15° són angles complementaris, tenim que:

$$\sin 15° = \cos 75° = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\cos 15° = \sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\operatorname{tg} 15° = \frac{1}{\operatorname{tg} 75°}=\frac{1}{\frac{\left(\sqrt{6}+\sqrt{2}\right)^2}{4}}=\frac{4}{\left(\sqrt{6}+\sqrt{2}\right)^2}$$

Finalment, comprovem aquests resultats calculant les raons trigonomètriques de 90° a partir dels angles de 15° i de 75°.

$$\sin 90° = \sin(75°+15°) = \sin 75°\cos 15° + \cos 75°\sin 15° =$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}\cdot\frac{\sqrt{6}+\sqrt{2}}{4}+\frac{\sqrt{6}-\sqrt{2}}{4}\cdot\frac{\sqrt{6}-\sqrt{2}}{4}=1$$

$$\cos 90° = \cos(75°+15°) = \cos 75°\cos 15° - \sin 75°\sin 15° =$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}\cdot\frac{\sqrt{6}+\sqrt{2}}{4}-\frac{\sqrt{6}+\sqrt{2}}{4}\cdot\frac{\sqrt{6}-\sqrt{2}}{4}=0$$

$$\operatorname{tg} 90° = \frac{\sin(75°+15°)}{\cos(75°+15°)}=\frac{1}{0}=\infty$$

27. Sigui $\operatorname{tg}\alpha=\frac{1}{3}$, tenim:

a) $\operatorname{tg}(180°-\alpha) = -\operatorname{tg}\alpha = -\frac{1}{3}$

b) $\operatorname{tg}(180°+\alpha) = \operatorname{tg}\alpha = \frac{1}{3}$

c) $\operatorname{tg}(360°-\alpha) = -\operatorname{tg}\alpha = -\frac{1}{3}$

28. Per començar, necessitem calcular el valor de cos α i el de tg α. Com que α està en el tercer quadrant, el valor del cosinus i de la tangent seran negatius. Hi apliquem el teorema fonamental de la trigonometria:

$$\sin^2\alpha+\cos^2\alpha=1 \Rightarrow \left(-\frac{1}{2}\right)^2+\cos^2\alpha=1$$

$$\Rightarrow \cos\alpha=-\sqrt{1-\frac{1}{4}}=-\frac{\sqrt{3}}{2}$$

$$\operatorname{tg}\alpha=\frac{\sin\alpha}{\cos\alpha}=\left(-\frac{1}{2}\right):\left(-\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{3}$$

a) $\sin(180°-\alpha) = \sin\alpha = -\frac{1}{2}$

b) $\sin(180°+\alpha) = -\sin\alpha = -\left(-\frac{1}{2}\right)=\frac{1}{2}$

c) $\cos(180°+\alpha) = -\cos\alpha = -\left(-\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{2}$

d) $\cos(180°-\alpha) = -\cos\alpha = -\left(-\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{2}$

e) $\operatorname{tg}(180°+\alpha) = \operatorname{tg}\alpha = \frac{\sqrt{3}}{3}$

f) $\operatorname{tg}(180°-\alpha) = -\operatorname{tg}\alpha = -\frac{\sqrt{3}}{3}$

29. L'angle d'$\frac{11\pi}{4}$ és superior a 2π; per tant, aquest angle fa més d'una volta a la circumferència. Si restem 2π a aquest angle (que és una volta sencera), tindrem un altre angle en el qual les seves raons trigonomètriques coincideixen amb l'angle d'$\frac{11\pi}{4}$.

És a dir, $\frac{11\pi}{4}-2\pi=\frac{3\pi}{4}$. Aquest angle és el que buscàvem.

30. a) $\sin 960° = \sin 240° = \sin(180°+60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

b) $\cos\left(\frac{41\pi}{6}\right)=\cos\left(\frac{5\pi}{6}\right)=\cos\left(\pi-\frac{\pi}{6}\right)=-\cos\left(\frac{\pi}{6}\right)=$
$=-\frac{\sqrt{3}}{2}$

c) $\operatorname{tg}\left(\frac{-9\pi}{2}\right)=\operatorname{tg}\left(\frac{3\pi}{2}\right)$. Aquesta tangent no està definida, ja que equival a l'angle de 270°.

31. Sabem que 180° < α < 270°. Aleshores:

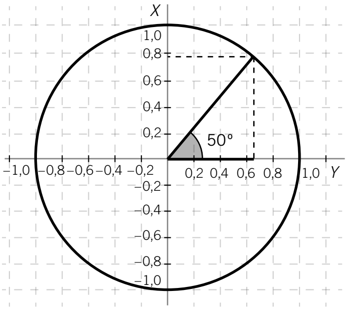
a) 0 < α – 180° < 90° → 0 > 180° – α > –90°
→ 270° < 180° – α < 360° → 4t quadrant

b) 360° < 180° + α < 450° → 0 < 180° + α < 90°
→ 1r quadrant

c) –180° < α – 360° < –90° → 90° < 360° – α < 180°
→ 2n quadrant

d) –180° > –α > –270° → 90° < – α < 180°
→ 2n quadrant

32. a)

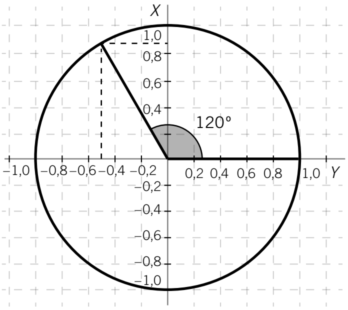


$$\sin 50° \approx 0{,}75$$

$$\cos 50° \approx 0{,}65$$

$$\text{tg } 50° = \frac{\sin 50°}{\cos 50°} \approx \frac{0{,}75}{0{,}65} = 1{,}15$$

b)

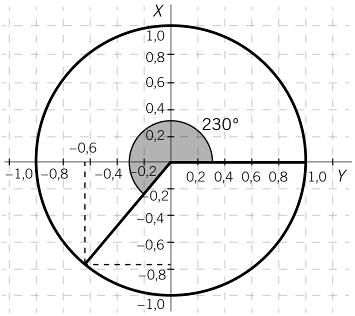


$$\sin 120° \approx 0{,}85$$

$$\cos 120° \approx -0{,}5$$

$$\text{tg } 120° = \frac{\sin 120°}{\cos 120°} \approx \frac{0{,}85}{-0{,}5} = -1{,}7$$

c)

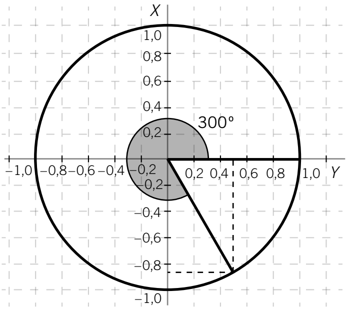


$$\sin 230° \approx -0{,}75$$

$$\cos 230° \approx -0{,}65$$

$$\text{tg } 230° = \frac{\sin 230°}{\cos 230°} \approx \frac{-0{,}75}{-0{,}65} = 1{,}15$$

d)



$$\sin 300° \approx -0{,}85$$

$$\cos 300° \approx 0{,}5$$

$$\text{tg } 300° = \frac{\sin 300°}{\cos 300°} \approx \frac{-0{,}85}{0{,}5} = -1{,}7$$

33. Sabem que α està en el tercer quadrant; per tant, el valor del cosinus serà negatiu i el valor de la tangent, positiu.

Utilitzem el teorema fonamental de la trigonometria per a calcular aquests valors:

$$\sin^2 \alpha + \cos^2 \alpha = 1 \Rightarrow \left(-\frac{1}{3}\right)^2 + \cos^2 \alpha = 1$$

$$\Rightarrow \cos \alpha = -\sqrt{1 - \frac{1}{9}} = -\frac{\sqrt{8}}{3}$$

$$\text{tg } \alpha = \frac{\sin \alpha}{\cos \alpha} = \left(-\frac{1}{3}\right) : \left(-\frac{\sqrt{8}}{3}\right) = \frac{1}{\sqrt{8}} = \frac{\sqrt{8}}{8}$$

34. Sabem que α es troba en el quart quadrant; per tant, el valor del sinus i de la tangent seran negatius.

Utilitzem el teorema fonamental de la trigonometria per a efectuar aquests càlculs:

$$\sin^2 \alpha + \cos^2 \alpha = 1 \Rightarrow \sin^2 \alpha + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$$\Rightarrow \sin \alpha = -\sqrt{1 - \frac{3}{4}} = -\frac{1}{2}$$

$$\text{tg } \alpha = \frac{\sin \alpha}{\cos \alpha} = \left(-\frac{1}{2}\right) : \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

35. a) L'angle de 225° pertany al tercer quadrant; així:

$$\sin 225° = \sin(180° + 45°) = -\sin 45° = -\frac{\sqrt{2}}{2} = -0{,}707$$

$$\cos 225° = \cos(180° + 45°) = -\cos 45° = -\frac{\sqrt{2}}{2} = -0{,}707$$

$$\text{tg } 225° = \text{tg}(180° + 45°) = \text{tg } 45° = 1$$

b) L'angle de 675° pertany al quart quadrant, ja que equival a l'angle de 315° (675° – 360°).

$$\sin 675° = \sin 315° = \sin(360° - 45°) = -\sin 45° = -0{,}707$$

$$\cos 675° = \cos 315° = \cos(360° - 45°) = \cos 45° = 0{,}707$$

$$\text{tg } 675° = \text{tg } 315° = \text{tg}(360° - 45°) = -\text{tg } 45° = -1$$

c) L'angle de –840° pertany al tercer quadrant, ja que equival a l'angle 240°.

$$\sin(-840°) = -\sin 840° = -\sin 120° = -\sin(180° - 60°) = $$
$$= -\sin 60° = -0{,}866$$

$$\cos(-840°) = \cos 840° = \cos 120° = \cos(180° - 60°) =$$
$$= -\cos 60° = -0{,}5$$

$$\text{tg}(-840°) = -\text{tg } 840° = -\text{tg } 120° = -\text{tg}(180° - 60°) =$$
$$= \text{tg } 60° = 1{,}732$$

4 RAONS TRIGONOMÈTRIQUES D'OPERACIONS AMB ANGLES

Pàgs. 117 i 118

36. En primer lloc, necessitem saber quin valor tenen cos 40° i tg 40°. Per a això, hi apliquem el teorema fonamental de la trigonometria:

$$\sin^2\alpha + \cos^2\alpha = 1 \Rightarrow (0{,}6428)^2 + \cos^2\alpha = 1 \Rightarrow$$
$$\Rightarrow \cos 40° = 0{,}766$$
$$\Rightarrow \text{tg}\,40° = \frac{\sin 40°}{\cos 40°} = \frac{0{,}6428}{0{,}766} = 0{,}847$$

Per a calcular les raons trigonomètriques de l'angle de 20°, hi apliquem les raons trigonomètriques de l'angle meitat. En aquest cas, les tres raons seran positives, ja que l'angle de 20° es troba en el primer quadrant.

$$\sin 20° = \sin\frac{40°}{2} = \sqrt{\frac{1-\cos 40°}{2}} = 0{,}342$$
$$\cos 20° = \cos\frac{40°}{2} = \sqrt{\frac{1+\cos 40°}{2}} = 0{,}9397$$
$$\text{tg}\,20° = \text{tg}\frac{40°}{2} = \sqrt{\frac{1-\cos 40°}{1+\cos 40°}} = 0{,}364$$

Per a calcular les raons trigonomètriques de l'angle de 80°, hi apliquem les raons trigonomètriques de l'angle doble. En aquest cas, les tres raons seran positives, ja que l'angle de 80° es troba en el primer quadrant.

$$\sin 80° = \sin(2\cdot 40°) = 2\cdot\sin 40°\cdot\cos 40° = 0{,}9848$$
$$\cos 80° = \cos(2\cdot 40°) = \cos^2 40° - \sin^2 40° = 0{,}1736$$
$$\text{tg}\,80° = \text{tg}(2\cdot 40°) = \frac{2\,\text{tg}\,40°}{1-\text{tg}^2\,40°} = 5{,}6713$$

37. Hem de descompondre cada angle en sumes o diferències d'angles que apareguin en la taula donada en l'enunciat.

a) 135° = 90° + 45°

$$\sin 135° = \sin(90°+45°) =$$
$$= \sin 90°\cdot\cos 45° + \cos 90°\cdot\sin 45° =$$
$$= 1\cdot\frac{\sqrt{2}}{2} + 0\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$
$$\cos 135° = \cos(90°+45°) =$$
$$= \cos 90°\cdot\cos 45° - \sin 90°\cdot\sin 45° =$$
$$= 0\cdot\frac{\sqrt{2}}{2} - 1\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$
$$\text{tg}\,135° = \frac{\sin 135°}{\cos 135°} = \frac{\sqrt{2}}{2} : \frac{-\sqrt{2}}{2} = -1$$

b) 210° = 180° + 30° = 2 · 90° + 30°

En primer lloc, calculem les raons trigonomètriques de 180°:

$$\sin(2\cdot 90°) = 2\cdot\sin 90°\cdot\cos 90° = 2\cdot 1\cdot 0 = 0$$
$$\cos(2\cdot 90°) = \cos^2 90° - \sin^2 90° = 0^2 - 1^2 = -1$$
$$\text{tg}(2\cdot 90°) = \frac{\sin(2\cdot 90°)}{\cos(2\cdot 90°)} = \frac{0}{-1} = 0$$

Ara, calculem les raons trigonomètriques de l'angle de 210°:

$$\sin 210° = \sin(180°+30°) =$$
$$= \sin 180°\cdot\cos 30° + \cos 180°\cdot\sin 30° =$$
$$= 0\cdot\frac{\sqrt{3}}{2} + (-1)\cdot\frac{1}{2} = -\frac{1}{2}$$
$$\cos 210° = \cos(180°+30°) =$$
$$= \cos 180°\cdot\cos 30° - \sin 180°\cdot\sin 30° =$$
$$= -1\cdot\frac{\sqrt{3}}{2} - 0\cdot\frac{1}{2} = -\frac{\sqrt{3}}{2}$$
$$\text{tg}\,210° = \text{tg}(180°+30°) = \frac{\text{tg}\,180° + \text{tg}\,30°}{1-\text{tg}\,180°\cdot\text{tg}\,30°} =$$
$$= \frac{0+\frac{\sqrt{3}}{3}}{1-0\cdot\frac{\sqrt{3}}{3}} = \frac{\sqrt{3}}{3}$$

c) 225° = 180° + 45° = 2 · 90° + 45°

Utilitzem els mateixos resultats de l'apartat (b) d'aquest exercici per a les raons trigonomètriques de l'angle de 180°. A partir d'aquestes, calculem les raons trigonomètriques de l'angle de 225°:

$$\sin 225° = \sin(180°+45°) =$$
$$= \sin 180°\cdot\cos 45° + \cos 180°\cdot\sin 45° =$$
$$= 0\cdot\frac{\sqrt{2}}{2} + (-1)\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$
$$\cos 225° = \cos(180°+45°) =$$
$$= \cos 180°\cdot\cos 45° - \sin 180°\cdot\sin 45° =$$
$$= -1\cdot\frac{\sqrt{2}}{2} - 0\cdot\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$
$$\text{tg}\,225° = \text{tg}(180°+45°) = \frac{\text{tg}\,180° + \text{tg}\,45°}{1-\text{tg}\,180°\cdot\text{tg}\,45°} =$$
$$= \frac{0+1}{1-0\cdot 1} = 1$$

d) 330° = 360° – 30° = 2 · 180° – 30°

L'angle de 360° és idèntic al de 0; per tant:

$$\sin 360° = 0 \qquad \cos 360° = 1 \qquad \text{tg}\,360° = 0$$

Ara, calculem les raons trigonomètriques de l'angle de 330° aplicant les raons trigonomètriques de la diferència d'angles:

$$\sin 330° = \sin(360°-30°) =$$
$$= \sin 360°\cdot\cos 30° - \cos 360°\cdot\sin 30° =$$
$$= 0\cdot\frac{\sqrt{3}}{2} - 1\cdot\frac{1}{2} = -\frac{1}{2}$$
$$\cos 330° = \cos(360°-30°) =$$
$$= \cos 360°\cdot\cos 30° + \sin 360°\cdot\sin 30° =$$
$$= 1\cdot\frac{\sqrt{3}}{2} + 0\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}$$
$$\text{tg}\,330° = \text{tg}(360°-30°) = \frac{\text{tg}\,360° - \text{tg}\,30°}{1+\text{tg}\,360°\cdot\text{tg}\,30°} =$$
$$= \frac{0-\frac{\sqrt{3}}{3}}{1+0\cdot\frac{\sqrt{3}}{3}} = -\frac{\sqrt{3}}{3}$$

e) 315° = 360° − 45° = 2 · 180° − 45°

Calculem les raons trigonomètriques de l'angle de 315° aplicant la diferència d'angles:

$$\sin 315° = \sin(360° - 45°) =$$
$$= \sin 360° \cdot \cos 45° - \cos 360° \cdot \sin 45° =$$
$$= 0 \cdot \frac{\sqrt{2}}{2} - 1 \cdot \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$

$$\cos 315° = \cos(360° - 45°) =$$
$$= \cos 360° \cdot \cos 45° + \sin 360° \cdot \sin 45° =$$
$$= 1 \cdot \frac{\sqrt{2}}{2} + 0 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

$$\operatorname{tg} 315° = \operatorname{tg}(360° - 45°) = \frac{\operatorname{tg} 360° - \operatorname{tg} 45°}{1 + \operatorname{tg} 360° \cdot \operatorname{tg} 45°} =$$
$$= \frac{0 - 1}{1 + 0 \cdot 1} = -1$$

38. L'angle de 75° és la suma de 45° i 30° dels quals coneixem les seves raons trigonomètriques; per tant:

$$\cos 75° = \cos(45° + 30°) = \cos 45° \cdot \cos 30° - \sin 45° \cdot \sin 30° =$$
$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

39. a) Necessitem saber els valors de sin α (positiu, ja que es troba en el primer quadrant), cos β (negatiu, ja que l'angle pertany al segon quadrant) i sin β (positiu).

$$\sin^2 \alpha + \cos^2 \alpha = 1 \Rightarrow \sin^2 \alpha + \left(\frac{2}{3}\right)^2 = 1 \Rightarrow \sin\alpha = 0{,}745$$

$$\left.\begin{array}{l} \operatorname{tg}\beta = \dfrac{\sin\beta}{\cos\beta} = -2{,}3 \\ \sin^2\beta + \cos^2\beta = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \sin\beta = -2{,}3\cos\beta \\ \sin\beta = \sqrt{1 - \cos^2\beta} \end{array}\right\} \Rightarrow$$
$$\Rightarrow -2{,}3\cos\beta = \sqrt{1 - \cos^2\beta} \Rightarrow \cos\beta = -0{,}399 \Rightarrow$$
$$\Rightarrow \sin\beta = 0{,}917$$

Ara, ja podem calcular cos (α + β):

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta =$$
$$= \frac{2}{3} \cdot (-0{,}399) - 0{,}745 \cdot 0{,}917 = -0{,}949$$

b) Necessitem saber el valor de tg α.

$$\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{0{,}745}{2/3} = 1{,}118$$

Ara, ja podem calcular tg (α + β):

$$\operatorname{tg}(\alpha + \beta) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha \cdot \operatorname{tg}\beta} = \frac{1{,}118 - 2{,}3}{1 - 1{,}118 \cdot (-2{,}3)} = -0{,}331$$

5 TRANSFORMACIÓ DE SUMES EN PRODUCTES

Pàg. 118

40. $$\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} =$$

$$= \frac{\dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\dfrac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} - \dfrac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}} =$$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}} = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{\operatorname{tg}\alpha - \operatorname{tg}\beta}$$

41. $\cos\alpha \cdot \cos(\alpha - \beta) + \sin\alpha \cdot \sin(\alpha - \beta) = \cos\beta$

$$\cos\alpha \cdot \cos(\alpha - \beta) + \sin\alpha \cdot \sin(\alpha - \beta) = \cos\alpha \cdot$$
$$(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + \sin\alpha \cdot (\sin\alpha\cos\beta - \cos\alpha\sin\beta) =$$
$$= \cos^2\alpha\cos\beta + \cancel{\cos\alpha\sin\alpha\sin\beta} + \sin^2\alpha\cos\beta -$$
$$- \cancel{\cos\alpha\sin\alpha\sin\beta} = \cos^2\alpha\cos\beta + \sin^2\alpha\cos\beta =$$
$$= \cos\beta \cdot (\cos^2\alpha + \sin^2\alpha) = \cos\beta$$

42. $$\frac{\cos(x - y)}{\cos(x + y)} = \frac{\cos x\cos y + \sin x\sin y}{\cos x\cos y - \sin x\sin y} =$$

$$= \frac{\dfrac{\cos x\cos y + \sin x\sin y}{\cos x\cos y}}{\dfrac{\cos x\cos y - \sin x\sin y}{\cos x\cos y}} = \frac{1 + \dfrac{\sin x\sin y}{\cos x\cos y}}{1 - \dfrac{\sin x\sin y}{\cos x\cos y}} =$$
$$= \frac{1 + \operatorname{tg} x\operatorname{tg} y}{1 - \operatorname{tg} x\operatorname{tg} y}$$

43. $$\frac{2\sin\alpha + 3}{2\operatorname{tg}\alpha + 3\sec\alpha} = \cos\alpha$$

$$\operatorname{tg}\alpha = \frac{\sin\alpha}{\cos\alpha},\ \sec\alpha = \frac{1}{\cos\alpha} \Rightarrow \frac{2\sin\alpha + 3}{2\operatorname{tg}\alpha + 3\sec\alpha} =$$
$$= \frac{2\sin\alpha + 3}{2 \cdot \dfrac{\sin\alpha}{\cos\alpha} + 3 \cdot \dfrac{1}{\cos\alpha}} = \frac{2\sin\alpha + 3}{\dfrac{2\sin\alpha + 3}{\cos\alpha}} =$$
$$= \frac{\cos\alpha \cdot (2\sin\alpha + 3)}{2\sin\alpha + 3} = \cos\alpha$$

44. a) $$\frac{\cos\alpha}{1 - \sin\alpha} + \frac{1 + \sin\alpha}{\cos\alpha} = \frac{\cos^2\alpha - \sin^2\alpha + 1}{(1 - \sin\alpha) \cdot \cos\alpha} =$$
$$= \frac{\cos 2\alpha + 1}{(1 - \sin\alpha) \cdot \cos\alpha} = \frac{\cos 2\alpha + 1}{\cos\alpha - \sin\alpha\cos\alpha} =$$
$$= \frac{\cos 2\alpha + 1}{\cos\alpha - 1/2\sin 2\alpha}$$

b) $$\sin(\alpha + \beta)\sin(\alpha - \beta) = (\sin\alpha\cos\beta + \cos\alpha\sin\beta) \cdot$$
$$\cdot (\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \sin^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta =$$
$$= \sin^2\alpha(1 - \sin^2\beta) + \cos^2\alpha\sin^2\beta =$$
$$= \sin^2\alpha - \sin^2\alpha\sin^2\beta + \cos^2\alpha\sin^2\beta =$$
$$= \sin^2\alpha - \sin^2\beta(\sin^2\alpha + \cos^2\alpha) = \sin^2\alpha - \sin^2\beta$$

c) $$2\left(\cos^2\frac{\alpha}{2} - \frac{1}{2}\cos\alpha\right) = 2\left(\left(\sqrt{\frac{1 + \cos\alpha}{2}}\right)^2 - \frac{1}{2}\cos\alpha\right) =$$
$$= 2\left(\frac{1 + \cos\alpha}{2} - \frac{1}{2}\cos\alpha\right) = 2\left(\frac{1 + \cancel{\cos\alpha} - \cancel{\cos\alpha}}{2}\right) = 1$$

d) $\dfrac{2\sin\alpha}{tg\, 2\alpha}+\dfrac{\sin^2\alpha}{\cos\alpha}=\dfrac{2\sin\alpha}{\dfrac{2\, tg\,\alpha}{1-tg^2\,\alpha}}+\dfrac{\sin^2\alpha}{\cos\alpha}=$

$=\dfrac{\sin\alpha\left(1-tg^2\,\alpha\right)}{tg\,\alpha}+\dfrac{\sin^2\alpha}{\cos\alpha}=$

$=\dfrac{\sin\alpha\left(1-tg^2\,\alpha\right)}{\dfrac{\sin\alpha}{\cos\alpha}}+\dfrac{\sin^2\alpha}{\cos\alpha}=$

$=\dfrac{\cancel{\sin\alpha}\cdot\left(1-tg^2\,\alpha\right)\cdot\cos\alpha}{\cancel{\sin\alpha}}+\dfrac{\sin^2\alpha}{\cos\alpha}=$

$=\dfrac{\left(1-tg^2\,\alpha\right)\cdot\cos^2\alpha+\sin^2\alpha}{\cos\alpha}=$

$=\dfrac{\cos^2\alpha-\cancel{\sin^2\alpha}+\cancel{\sin^2\alpha}}{\cos\alpha}=$

$=\dfrac{\cos^{\cancel{2}}\alpha}{\cancel{\cos\alpha}}=\cos\alpha$

45. a) $\dfrac{\cos 2x}{\sin x+\cos x}=\dfrac{\cos^2 x-\sin^2 x}{\sin x+\cos x}=$

$=\dfrac{(\cos x-\sin x)\left(\cancel{\cos x+\sin x}\right)}{\cancel{\sin x+\cos x}}=\cos x-\sin x$

b) $\left(\cos^3 x-\cos 3x\right) tg\, x=$

$=\left(\cos^3 x-\cos^3 x+3\sin^2 x\cos x\right) tg\, x=$

$=\left(3\sin^2 x\,\cancel{\cos x}\right)\dfrac{\sin x}{\cancel{\cos x}}=3\sin^3 x$

46. a) $tg^2\,\alpha=\dfrac{\sin^2\alpha}{\cos^2\alpha}=\dfrac{\sin^2\alpha}{1-\sin^2\alpha}$

b) $cosec^4\,\alpha-1=\dfrac{1}{\sin^4\alpha}-1=\left(\dfrac{1}{\sin^2\alpha}-1\right)\left(\dfrac{1}{\sin^2\alpha}+1\right)=$

$=\left(1+cotg^2\,\alpha-1\right)\left(1+cotg^2\,\alpha+1\right)=cotg^2\,\alpha\left(2+cotg^2\,\alpha\right)=$

$=2\,cotg^2\,\alpha+cotg^4\,\alpha$

c) $\left(\sin\alpha+\cos\alpha\right)^2=\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha=$

$=1+2\sin\alpha\cos\alpha=1+2\sin\alpha\dfrac{1}{\sec\alpha}=1+2\dfrac{\sin\alpha}{\sec\alpha}$

47. $\dfrac{\cos^4 x-\sin^4 x}{1-tg^4\, x}=\dfrac{\left(\cos^2 x-\sin^2 x\right)\left(\cos^2 x+\sin^2 x\right)}{\left(1-tg^2\, x\right)\left(1+tg^2\, x\right)}=$

$=\dfrac{\cos^2 x-\sin^2 x}{\left(1-tg^2\, x\right)\dfrac{1}{\cos^2 x}}=\dfrac{\cos^2 x\left(\cos^2 x-\sin^2 x\right)}{\left(1-\dfrac{\sin^2 x}{\cos^2 x}\right)}=$

$=\dfrac{\cos^2 x\left(\cos^2 x-\sin^2 x\right)}{\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x}}=\dfrac{\cos^4 x\left(\cancel{\cos^2 x-\sin^2 x}\right)}{\cancel{\cos^2 x-\sin^2 x}}=$

$=\cos^4 x$

48. $\left(\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}}-\sqrt{\dfrac{1-\sin\alpha}{1+\sin\alpha}}\right)\cdot\left(\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}-\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}\right)=$

$=\left(\sqrt{\dfrac{(1+\sin\alpha)(1+\sin\alpha)}{(1-\sin\alpha)(1+\sin\alpha)}}-\sqrt{\dfrac{(1-\sin\alpha)(1-\sin\alpha)}{(1+\sin\alpha)(1-\sin\alpha)}}\right)\cdot$

$\cdot\left(\sqrt{\dfrac{(1+\cos\alpha)(1+\cos\alpha)}{(1-\cos\alpha)(1+\cos\alpha)}}-\sqrt{\dfrac{(1-\cos\alpha)(1-\cos\alpha)}{(1+\cos\alpha)(1-\cos\alpha)}}\right)=$

$=\left(\sqrt{\dfrac{(1+\sin\alpha)^2}{\cos^2\alpha}}-\sqrt{\dfrac{(1-\sin\alpha)^2}{\cos^2\alpha}}\right)\cdot$

$\cdot\left(\sqrt{\dfrac{(1+\cos\alpha)^2}{\sin^2\alpha}}-\sqrt{\dfrac{(1-\cos\alpha)^2}{\sin^2\alpha}}\right)=$

$=\left(\dfrac{1+\sin\alpha}{\cos\alpha}-\dfrac{1-\sin\alpha}{\cos\alpha}\right)\cdot\left(\dfrac{1+\cos\alpha}{\sin\alpha}-\dfrac{1-\cos\alpha}{\sin\alpha}\right)=$

$=\left(\dfrac{2\sin\alpha}{\cos\alpha}\right)\cdot\left(\dfrac{2\cos\alpha}{\sin\alpha}\right)=4$

49. $\dfrac{\sin^2\alpha\left(1+cotg^2\,\alpha\right)+\cos^2\alpha\left(1+tg^2\,\alpha\right)}{\cos^2\alpha\left(\sec^2\alpha-tg^2\,\alpha\right)+\sin^2\alpha\left(cosec^2\,\alpha-cotg^2\,\alpha\right)}=$

$=\dfrac{\cancel{\sin^2\alpha}\cdot\dfrac{1}{\cancel{\sin^2\alpha}}+\cancel{\cos^2\alpha}\cdot\dfrac{1}{\cancel{\cos^2\alpha}}}{\cos^2\alpha\cdot 1+\sin^2\alpha\cdot 1}=\dfrac{1+1}{1}=2$

50. a) $\dfrac{\sin x+\cos x}{\sin x}=\dfrac{\sin x}{\sin x}+\dfrac{\cos x}{\sin x}=1+cotg\, x=1+\dfrac{1}{tg\, x}$

b) $\dfrac{\sec x}{tg\, x+cotg\, x}=\dfrac{1/\cos x}{tg\, x+\dfrac{1}{tg\, x}}=\dfrac{1/\cos x}{\dfrac{tg^2\, x+1}{tg\, x}}=$

$=\dfrac{tg\, x}{\cos x\left(tg^2\, x+1\right)}=\dfrac{\sin x/\cos x}{\cos x\left(\dfrac{\sin^2 x}{\cos^2 x}+1\right)}=$

$=\dfrac{\sin x/\cos x}{\dfrac{\sin^2 x}{\cos x}+\cos x}=\dfrac{\sin x/\cos x}{\dfrac{\sin^2 x+\cos^2 x}{\cos x}}=$

$=\dfrac{\sin x/\cos x}{1/\cos x}=\sin x$

c) $\dfrac{\sin x+tg\, x}{1+\sec x}=\dfrac{\sin x+\dfrac{\sin x}{\cos x}}{1+\dfrac{1}{\cos x}}=\dfrac{\dfrac{\sin x(\cos x+1)}{\cos x}}{\dfrac{\cos x+1}{\cos x}}=$

$=\dfrac{\sin x\left(\cancel{\cos x+1}\right)}{\cancel{\cos x+1}}=\sin x$

51. a) $\cos^2 135°-\dfrac{1}{2}\, tg\ 225°\cdot\sin^2 300°-cotg\ 315°=$

$=\cos^2(180°-45°)-\dfrac{1}{2}\, tg\,(180°+45°)\sin^2(360°-60°)-$

$-cotg\,(360°-45°)=\cos^2 45°-\dfrac{1}{2}\, tg\ 45°\cdot\sin^2 60°-$

$-\dfrac{1}{-tg\, 45°}=\left(\dfrac{\sqrt{2}}{2}\right)^2-\dfrac{1}{2}\cdot 1\cdot\left(\dfrac{\sqrt{3}}{2}\right)^2+\dfrac{1}{1}=\dfrac{9}{8}$

b) $\sin 45° - \text{tg}^2 30° \cdot \text{cotg } 240° \cdot \sec 150° =$

$= \sin 45° - \text{tg}^2 30° \cdot \text{cotg}(180° + 60°) \cdot \sec(180° - 30°) =$

$= \sin 45° - \text{tg}^2 30° \cdot \dfrac{1}{\text{tg } 60°} \cdot \dfrac{1}{-\cos 30°} =$

$= \dfrac{\sqrt{2}}{2} - \left(\dfrac{\sqrt{3}}{3}\right)^2 \cdot \dfrac{1}{\sqrt{3}} \cdot \dfrac{1}{-\dfrac{\sqrt{3}}{2}} = \dfrac{9\sqrt{2}+4}{18}$

52. a) $\cos(x+y)\cos(x-y) = \dfrac{1}{2}(\cos 2x + \cos 2y) =$

$= \dfrac{1}{2}(\cos^2 x - \sin^2 x + \cos^2 y - \sin^2 y) =$

$= \dfrac{1}{2}(\cos^2 x - (1-\cos^2 x) + (1-\sin^2 y) - \sin^2 y) =$

$= \dfrac{1}{2}(2\cos^2 x - 2\sin^2 y) = \cos^2 x - \sin^2 y$

b) $\sin^2\left(\dfrac{x+y}{2}\right) - \sin^2\left(\dfrac{x-y}{2}\right) =$

$= \dfrac{1-\cos(x+y)}{2} - \dfrac{1-\cos(x-y)}{2} =$

$= \dfrac{\cos(x-y) - \cos(x+y)}{2} =$

$= \dfrac{\cos x \cdot \cos y + \sin x \cdot \sin y - \cos x \cdot \cos y + \sin x \cdot \sin y}{2} =$

$= \dfrac{2\sin x \cdot \sin y}{2} = \sin x \cdot \sin y$

c) $\cos^2\left(\dfrac{x-y}{2}\right) - \cos^2\left(\dfrac{x+y}{2}\right) =$

$= \dfrac{1+\cos(x-y)}{2} - \dfrac{1+\cos(x+y)}{2} =$

$= \dfrac{\cos(x-y) - \cos(x+y)}{2} =$

$= \dfrac{\cos x \cdot \cos y + \sin x \cdot \sin y - \cos x \cdot \cos y + \sin x \cdot \sin y}{2} =$

$= \dfrac{2\sin x \cdot \sin y}{2} = \sin x \cdot \sin y$

53. Calculem les raons trigonomètriques dels angles α i β.

— L'angle α pertany al primer quadrant; per tant, els valors de les raons trigonomètriques seran positius.

$\sin\alpha = \dfrac{1}{4}$; $\sin^2\alpha + \cos^2\alpha = 1 \Rightarrow \left(\dfrac{1}{4}\right)^2 + \cos^2\alpha = 1$

$\Rightarrow \cos\alpha = \sqrt{1 - \dfrac{1}{16}} = \dfrac{\sqrt{15}}{4} \Rightarrow$

$\Rightarrow \text{tg}\,\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\dfrac{1}{4}}{\dfrac{\sqrt{15}}{4}} = \dfrac{\sqrt{15}}{15}$

— L'angle β pertany al tercer quadrant; per tant, el sinus i el cosinus seran negatius, mentre que la tangent serà positiva.

$\cos\beta = -\dfrac{1}{3}$; $\sin^2\beta + \cos^2\beta = 1 \Rightarrow \sin^2\beta + \left(-\dfrac{1}{3}\right)^2 = 1$

$\Rightarrow \sin\beta = -\sqrt{1 - \dfrac{1}{9}} = -\dfrac{\sqrt{8}}{3} \Rightarrow \text{tg}\,\beta = \dfrac{\sin\beta}{\cos\beta} =$

$= \dfrac{-\dfrac{\sqrt{8}}{3}}{-\dfrac{1}{3}} = \sqrt{8}$

Ara, ja podem calcular les raons trigonomètriques de l'angle $\alpha + \beta$.

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta =$

$= \dfrac{1}{4} \cdot \left(-\dfrac{1}{3}\right) + \dfrac{\sqrt{15}}{4} \cdot \left(-\dfrac{\sqrt{8}}{3}\right) = \dfrac{-1+\sqrt{120}}{12}$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta =$

$= \dfrac{\sqrt{15}}{4} \cdot \left(-\dfrac{1}{3}\right) - \dfrac{1}{4} \cdot \left(-\dfrac{\sqrt{8}}{3}\right) = \dfrac{-\sqrt{15}+\sqrt{8}}{12}$

$\text{tg}(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\dfrac{-1+\sqrt{120}}{12}}{\dfrac{-\sqrt{15}+\sqrt{8}}{12}} =$

$= \dfrac{-2\sqrt{2} - \sqrt{15} + 2\sqrt{30}\sqrt{15} + 4\sqrt{2}\sqrt{30}}{-7}$

6 EQUACIONS TRIGONOMÈTRIQUES I SISTEMES

Pàgs. 119 i 120

54. a) $\sin x = \dfrac{\sqrt{3}}{2} \Rightarrow x = \text{arc sin}\left(\dfrac{\sqrt{3}}{2}\right) \Rightarrow x_1 = 60°,\ x_2 = 120°$

b) $\cos x = \dfrac{1}{\sqrt{2}} \Rightarrow x = \text{arc cos}\left(\dfrac{1}{\sqrt{2}}\right) \Rightarrow x_1 = 45°,\ x_2 = 315°$

c) $\text{tg}\,x = \dfrac{1}{\sqrt{3}} \Rightarrow x = \text{arc tg}\left(\dfrac{1}{\sqrt{3}}\right) \Rightarrow x_1 = 30°,\ x_2 = 210°$

d) $\cos(x+45°) = \dfrac{1}{2} \Rightarrow x + 45° = \text{arc cos}\left(\dfrac{1}{2}\right) \Rightarrow$

$\Rightarrow \begin{cases} x + 45° = 60° \\ x + 45° = 300° \end{cases} \Rightarrow x_1 = 15°,\ x_2 = 255°$

e) $\cos x \cdot \text{cotg}\,x = \dfrac{3}{2} \Rightarrow \cos x \cdot \dfrac{\cos x}{\sin x} = \dfrac{3}{2} \Rightarrow \dfrac{\cos^2 x}{\sin x} = \dfrac{3}{2}$

$\dfrac{1-\sin^2 x}{\sin x} = \dfrac{3}{2} \Rightarrow 2\sin^2 x + 3\sin x - 2 = 0$

$\Rightarrow \sin x = \dfrac{-3 \pm 5}{4} \Rightarrow \sin x = \dfrac{1}{2} \Rightarrow x = \text{arc sin}\left(\dfrac{1}{2}\right)$

$\Rightarrow x_1 = 30°,\ x_2 = 150°$

f) $\sin x + \cos x = \sqrt{2} \Rightarrow (\sin x + \cos x)^2 = 2$

$\Rightarrow \sin^2 x + \cos^2 x + 2\sin x\cos x = 2 \Rightarrow 2\sin x\cos x = 1$

$\Rightarrow \sin 2x = 1 \Rightarrow 2x = \text{arc sin}(1) \Rightarrow 2x = 90° \Rightarrow x = 45°$

g) $\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x = 0 \Rightarrow \cos(x + 60°) = 0$

$\Rightarrow x + 60° = \text{arc cos}(0) \Rightarrow \begin{cases} x + 60° = 90° \Rightarrow x_1 = 30° \\ x + 60° = 270° \Rightarrow x_2 = 210° \end{cases}$

55. $\sin 3x = 0 \Rightarrow 3x = \text{arc sin}(0) \Rightarrow 3x = 180° \cdot k \Rightarrow$

$\Rightarrow x = 60° \cdot k$

56. a) $\sin(2x) \cdot \cos x = 6 \cdot \sin^3 x \Rightarrow$

$\Rightarrow 2 \sin x \cos x \cos x = 6 \cdot \sin^3 x$

$\Rightarrow \sin x(\cos^2 x - 3\sin^2 x) = 0 \Rightarrow \begin{cases} \sin x = 0 \\ \cos^2 x - 3\sin^2 x = 0 \end{cases}$

$\Rightarrow \begin{cases} x = \text{arc sin}(0) \Rightarrow x_1 = 0,\ x_2 = 180° \\ \cos^2 x = 3\sin^2 x \Rightarrow \text{tg}^2 x = \frac{1}{3} \Rightarrow \text{tg}\, x = \pm\frac{\sqrt{3}}{3} \Rightarrow \end{cases}$

$\Rightarrow \begin{cases} x = \text{arc tg}\left(\frac{\sqrt{3}}{3}\right) \Rightarrow x_3 = 30°,\ x_4 = 210° \\ x = \text{arc tg}\left(-\frac{\sqrt{3}}{3}\right) \Rightarrow x_5 = 150°,\ x_6 = 330° \end{cases}$

b) $\sin(3x) \cdot \cos(2x) = 1 + \sin(2x) \cdot \cos(3x)$

$\Rightarrow \frac{1}{2}(\sin(5x) + \sin x) = 1 + \frac{1}{2}(\sin(5x) + \sin(-x))$

$\Rightarrow \frac{1}{2}\sin x = 1 - \frac{1}{2}\sin x \Rightarrow \sin x = 1 \Rightarrow x = \text{arc sin}(1)$

$\Rightarrow x = 90°$

c) $\cos x = 2 \cdot \sin x \Rightarrow \frac{\sin x}{\cos x} = \frac{1}{2} \Rightarrow \text{tg}\, x = \frac{1}{2}$

$\Rightarrow x = \text{arc tg}\left(\frac{1}{2}\right) \Rightarrow \begin{cases} x_1 = 26{,}57° \\ x_2 = 206{,}57° \end{cases}$

d) $(\text{tg}\, x - 1) \cdot (4 \cdot \sin^2 x - 3) = 0 \Rightarrow \begin{cases} \text{tg}\, x - 1 = 0 \\ 4 \cdot \sin^2 x - 3 = 0 \end{cases}$

$\Rightarrow \begin{cases} x = \text{arc tg}(1) \Rightarrow x_1 = 45°,\ x_2 = 225° \\ \sin^2 x = \frac{3}{4} \Rightarrow x = \text{arc sin}\left(\pm\sqrt{\frac{3}{4}}\right) \end{cases}$

$\Rightarrow x_3 = 60°,\ x_4 = 120°, x_5 = 240°,\ x_6 = 300°$

e) $\sin x \cdot \cos^2 x - \frac{1}{2}\sin x + \frac{1}{2}\cos^2 x = \frac{1}{4}$

$\sin x \cdot \left(\cos^2 x - \frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2}\cos^2 x = \frac{1}{2}\left(\frac{1}{2} - \cos^2 x\right)$

$\left(\sin x + \frac{1}{2}\right) \cdot \left(\cos^2 x - \frac{1}{2}\right) = 0 \rightarrow \begin{cases} \sin x + \frac{1}{2} = 0 \\ \cos^2 x - \frac{1}{2} = 0 \end{cases}$

$x = \text{arc sin}\left(-\frac{1}{2}\right) \rightarrow x_1 = 210°;\ x_2 = 330°$

$x = \text{arc cos}\left(\pm\sqrt{\frac{1}{2}}\right) \rightarrow \begin{matrix} x_3 = 45; x_4 = 135; \\ x_5 = 225;\ x_6 = 315 \end{matrix}$

57. a) $4 \sin x \cdot \cos x = \sqrt{3} \rightarrow 4 \sin x \cdot \sqrt{1 - \sin^2 x} = \sqrt{3}$

$16 \sin^2 x \cdot (1 - \sin^2 x) = 3 \rightarrow 16 \sin^4 x - 16 \sin^2 x + 3 = 0$

Efectuem el canvi de variable $\sin^2 x = p$ i resolem l'equació.

$$16p^2 - 16p + 3 = 0 \rightarrow \begin{cases} p_1 = \frac{3}{4} \\ p_2 = \frac{1}{4} \end{cases}$$

$$\sin x = \pm\sqrt{\frac{3}{4}} \Rightarrow x = \frac{\pi}{3} + \pi k$$

$$\sin x = \pm\sqrt{\frac{1}{2}} \Rightarrow x = \frac{\pi}{6} + \pi k$$

b) $\frac{2}{\sin x} = \frac{3}{\cos^2 x} \Rightarrow 2\cos^2 x = 3 \sin x$

$2(1 - \sin^2 x) = 3 \sin x$

$\Rightarrow -2\sin^2 x - 3\sin x + 2 = 0 \Rightarrow \sin x = \frac{3 \pm 5}{-4} = \frac{1}{2}$

$\Rightarrow x = \text{arc sin}\left(\frac{1}{2}\right) \Rightarrow$

$\Rightarrow x_1 = 30° + 360° \cdot k,\ x_2 = 150° + 360° \cdot k$

$\sin x = -2$ no és solució, ja que $-1 \le \sin x \le 1$

Expressem els resultats en radians:

$$x_1 = 30° + 360° \cdot k = 30\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} + 2\pi \cdot k = \frac{\pi}{6} + 2\pi \cdot k$$

$$x_2 = 150° + 360° \cdot k = 150\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} + 2\pi \cdot k = \frac{5\pi}{6} + 2\pi \cdot k$$

c) $2\cos x + 3\,\text{tg}\, x = 0 \Rightarrow 2\cos x + 3 \cdot \frac{\sin x}{\cos x} = 0$

$\Rightarrow 2\cos^2 x + 3\sin x = 0 \Rightarrow 2(1 - \sin^2 x) + 3 \sin x = 0$

$\Rightarrow -2\sin^2 x + 3\sin x + 2 = 0 \Rightarrow \sin x = \frac{-3 \pm 5}{-4} = -\frac{1}{2}$

$\Rightarrow x = \text{arc sin}\left(-\frac{1}{2}\right) \Rightarrow$

$\Rightarrow x_1 = -30° + 360° \cdot k,\ x_2 = 210° + 360° \cdot k$

$\sin x = 2$ no és solució, ja que $-1 \le \sin x \le 1$

Expressem els resultats en radians:

$$x_1 = -30° + 360° \cdot k = -30\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} + 2\pi \cdot k = \frac{-\pi}{6} + 2\pi \cdot k$$

$$x_2 = 210° + 360° \cdot k = 210\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} + 2\pi \cdot k = \frac{7\pi}{6} + 2\pi \cdot k$$

58. a) $\left.\begin{matrix} \sin x + \cos y = 1 \\ 2x + 2y = 180° \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \sin x + \cos y = 1 \\ x = 90° - y \end{matrix}\right\} \Rightarrow$

$\Rightarrow \sin(90° - y) + \cos y = 1 \Rightarrow \cos y + \cos y = 1$

$\Rightarrow \cos y = \frac{1}{2} \Rightarrow y = \text{arccos}\left(\frac{1}{2}\right) \Rightarrow \begin{cases} y_1 = 60° \\ y_2 = -60° = 300° \end{cases}$

$\Rightarrow x_1 = 90° - 60° = 30°,\ x_2 = 90° - (-60°) = 150°$

b) $\left.\begin{array}{l}\cos x + \cos y = \sqrt{2}\\ x + y = 90^\circ\end{array}\right\} \Rightarrow \left.\begin{array}{l}2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2} = \sqrt{2}\\ x + y = 90^\circ\end{array}\right\} \Rightarrow$

$\Rightarrow 2\cos\dfrac{90^\circ}{2}\cos\dfrac{x-y}{2} = \sqrt{2} \Rightarrow$

$\Rightarrow 2\cos 45^\circ \cos\dfrac{x-y}{2} = \sqrt{2} \Rightarrow 2\dfrac{\sqrt{2}}{2}\cos\dfrac{x-y}{2} = \sqrt{2}$

$\Rightarrow \cos\dfrac{x-y}{2} = 1 \Rightarrow \dfrac{x-y}{2} = \text{arc}\cos(1) = 0$

$\Rightarrow x = y \Rightarrow x + y = x + x = 90^\circ \Rightarrow x = 45^\circ,\ y = 45^\circ$

c) $\left.\begin{array}{l}x + \sin^2 y = 2\\ x + \cos^2 y = 1\end{array}\right\}$

$2x + \sin^2 y + \cos^2 y = 3 \Rightarrow 2x = 2 \Rightarrow x = 1$

$\Rightarrow \left.\begin{array}{l}\sin^2 y = 1\\ \cos^2 y = 0\end{array}\right\} \Rightarrow \left.\begin{array}{l}y = \text{arc}\sin(\pm 1)\\ y = \text{arc}\cos(0)\end{array}\right\} \Rightarrow y_1 = 90^\circ,\ y_2 = 270^\circ$

59. $\left.\begin{array}{l}4y \cdot \sin x \cos x = 3\\ 2y \cdot \cos 2x = \sqrt{3}\end{array}\right\} \to \left.\begin{array}{l}2y\cdot = \sqrt{2}\,2x = 3\\ 2y \cdot \cos 2x = \sqrt{3}\end{array}\right\}$

Si dividim les dues equacions, obtenim:

$\text{tg}\, 2x = \sqrt{3} \to x = \dfrac{1}{2}\text{arc tg}\sqrt{3} \to \begin{cases} x_1 = \dfrac{2\pi}{3} + \pi k\\ x_2 = \dfrac{7\pi}{6} + \pi k\end{cases}$

Si elevem al quadrat les dues equacions del sistema i les sumem, obtenim:

$$4y^2(\sin^2 2x + \cos^2 2x) = 9 + 3 \to 4y^2 = 12$$

$$y = \pm\sqrt{3}$$

60. $2\sin^2 x - \sin x = 1$

$\sin x = \dfrac{1 \pm \sqrt{(-1)^2 - 4\cdot 2\cdot(-1)}}{2\cdot 2} \to \begin{cases}\sin x = 1\\ \sin x = -\dfrac{1}{2}\end{cases}$

$x_1 = \text{arc}\sin(1) = 90^\circ$

$x = \text{arc}\sin\left(-\dfrac{1}{2}\right) \to \begin{cases}x_2 = 210^\circ\\ x_3 = 330^\circ\end{cases}$

61. a) $\sin(45^\circ + x) - \sqrt{2}\sin x = 0$

$\Rightarrow \sin 45^\circ \cos x + \cos 45^\circ \sin x - \sqrt{2}\sin x = 0$

$\Rightarrow \dfrac{\sqrt{2}}{2}\cos x + \dfrac{\sqrt{2}}{2}\sin x - \sqrt{2}\sin x = 0 \Rightarrow \cos x = \sin x$

$\Rightarrow \dfrac{\sin x}{\cos x} = 1 \Rightarrow \text{tg}\, x = 1 \Rightarrow x = \text{arc tg}(1) \Rightarrow$

$\Rightarrow x_1 = 45^\circ, x_2 = 225^\circ$

b) $\sin(30^\circ - x) - \sin(60^\circ - x) = 0$

$\sin 30^\circ \cos x - \cos 30^\circ \sin x - \sin 60^\circ \cos x +$
$+ \cos 60^\circ \sin x = 0$

$\dfrac{1}{2}\cos x - \dfrac{\sqrt{3}}{2}\sin x - \dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x = 0$

$\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\right)(\cos x + \sin x) = 0 \to \sin x = -\cos x$

$\text{tg}\, x = -1 \to x = \text{arc tg}(-1)\begin{cases}x_1 = 135^\circ\\ x_2 = 315^\circ\end{cases}$

62. a) $\left.\begin{array}{l}\cos x + \cos y = 1\\ \cos(x + y) = 1\end{array}\right\} \Rightarrow \left.\begin{array}{l}\cos x + \cos y = 1\\ x + y = \text{arc}\cos(1)\end{array}\right\}$

$\Rightarrow \left.\begin{array}{l}\cos x + \cos y = 1\\ x + y = 0\end{array}\right\}$

$\Rightarrow \cos x + \cos(-x) = 1 \Rightarrow \cos x = \dfrac{1}{2} \Rightarrow \begin{cases}x_1 = \dfrac{5\pi}{3} + 2\pi\cdot k\\ x_2 = \dfrac{\pi}{3} + 2\pi\cdot k\end{cases}$

$\Rightarrow y_1 = \dfrac{\pi}{3} + 2\pi\cdot k,\ y_2 = \dfrac{5\pi}{3} + 2\pi\cdot k$

b) $\left.\begin{array}{l}x + y = \dfrac{\pi}{2}\\ \sin x + \sin y = \dfrac{\sqrt{6}}{2}\end{array}\right\} \Rightarrow$

$\Rightarrow \left.\begin{array}{l}x + y = \dfrac{\pi}{2}\\ 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2} = \dfrac{\sqrt{6}}{2}\end{array}\right\} \Rightarrow$

$\Rightarrow 2\sin\left(\dfrac{\pi}{4}\right)\cos\dfrac{x-y}{2} = \dfrac{\sqrt{6}}{2} \Rightarrow 2\cdot\dfrac{\sqrt{2}}{2}\cos\dfrac{x-y}{2} = \dfrac{\sqrt{6}}{2}$

$\Rightarrow \cos\dfrac{x-y}{2} = \dfrac{\sqrt{3}}{2} \Rightarrow \dfrac{x-y}{2} = \dfrac{\pi}{6} \Rightarrow x - y = \dfrac{\pi}{3} \Rightarrow$

$\Rightarrow \begin{cases}x - y = \dfrac{\pi}{3}\\ x + y = \dfrac{\pi}{2}\end{cases} \Rightarrow x = \dfrac{5\pi}{12} + 2\pi\cdot k,\ y = \dfrac{\pi}{12} + 2\pi\cdot k$

Observem que en l'equació es poden intercanviar *x* i *y*; en aquest cas, podem intercanviar els valors de *x* i de *y*, ja que el resultat no canviarà. Aleshores:

$$x_1 = \dfrac{\pi}{12} + 2\pi\cdot k,\ y_1 = \dfrac{5\pi}{12} + 2\pi\cdot k$$

$$x_2 = \dfrac{5\pi}{12} + 2\pi\cdot k,\ y_2 = \dfrac{\pi}{12} + 2\pi\cdot k$$

63. Sigui $s = 2x$ i $t = 2y$. Aleshores, tenim:

$\left.\begin{array}{l}\sin(2x) + \cos(3y) = 1\\ 2\sin(2x) + 4\cos(3y) = 3\end{array}\right\} \Rightarrow \left.\begin{array}{l}\sin s + \cos t = 1\\ 2\sin s + 4\cos t = 3\end{array}\right\} \Rightarrow$

$\Rightarrow \left.\begin{array}{l}\sin s + \cos t = 1\\ \sin s + 2\cos t = \dfrac{3}{2}\end{array}\right\} \Rightarrow \cos t = \dfrac{1}{2} \Rightarrow t = \text{arc}\cos\left(\dfrac{1}{2}\right) \Rightarrow t = 60^\circ$

$\Rightarrow \sin s + \cos 60^\circ = 1 \Rightarrow \sin s = \dfrac{1}{2} \Rightarrow s = \text{arc}\sin\left(\dfrac{1}{2}\right) \Rightarrow s = 30^\circ$

$\Rightarrow \begin{cases}2x = 30^\circ \Rightarrow x = 15^\circ\\ 3y = 60^\circ \Rightarrow y = 20^\circ\end{cases}$

Expressem els resultats en radians:

$x = 15^\circ \cdot \dfrac{2\pi \text{ rad}}{360^\circ} = \dfrac{\pi}{12} \text{ rad},\ y = 20^\circ \cdot \dfrac{2\pi \text{ rad}}{360^\circ} = \dfrac{\pi}{9} \text{ rad}$

64. $\left.\begin{array}{l}\sin x + \sin y = \dfrac{\sqrt{6}}{2}\\ \cos x + \cos y = \dfrac{\sqrt{2}}{2}\end{array}\right\} \Rightarrow \left.\begin{array}{l}2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2} = \dfrac{\sqrt{6}}{2}\\ 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2} = \dfrac{\sqrt{2}}{2}\end{array}\right\} \Rightarrow$

$$\Rightarrow \frac{\cancel{2}\sin\dfrac{x+y}{2}\cancel{\cos\dfrac{x-y}{2}}}{\cancel{2}\cos\dfrac{x+y}{2}\cancel{\cos\dfrac{x-y}{2}}} = \frac{\dfrac{\sqrt{6}}{2}}{\dfrac{\sqrt{2}}{2}} \Rightarrow \text{tg}\,\frac{x+y}{2} = \sqrt{3}$$

$$\Rightarrow \frac{x+y}{2} = \text{arc tg}\left(\sqrt{3}\right) \Rightarrow x + y = 120°$$

$$\Rightarrow \left.\begin{array}{l}2\sin\dfrac{120°}{2}\cos\dfrac{x-y}{2} = \dfrac{\sqrt{6}}{2}\\ 2\cos\dfrac{120°}{2}\cos\dfrac{x-y}{2} = \dfrac{\sqrt{2}}{2}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}2\cdot\dfrac{\sqrt{3}}{2}\cos\dfrac{x-y}{2} = \dfrac{\sqrt{6}}{2}\\ 2\cdot\dfrac{1}{2}\cos 60°\cos\dfrac{x-y}{2} = \dfrac{\sqrt{2}}{2}\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}\cos\dfrac{x-y}{2} = \dfrac{\sqrt{2}}{2}\\ \cos\dfrac{x-y}{2} = \dfrac{\sqrt{2}}{2}\end{array}\right\} \Rightarrow \frac{x-y}{2} = 45° \Rightarrow x - y = 90°$$

$$\Rightarrow \left.\begin{array}{l}x + y = 120°\\ x - y = 90°\end{array}\right\} \Rightarrow x = 105°,\ y = 15°$$

Com que es tracta d'equacions simètriques per a x i y, podem intercanviar els valors de x i de y, ja que el resultat no canviarà. Aleshores: $x = 105°$, $y = 15°$, $x = 15°$, $y = 105°$.

Expressem els resultats en radians:

$$x = 105\cancel{°}\cdot\frac{2\pi\text{ rad}}{360\cancel{°}} = \frac{7\pi}{12}\text{ rad},\ y = 15\cancel{°}\cdot\frac{2\pi\text{ rad}}{360\cancel{°}} = \frac{\pi}{12}\text{ rad}$$

65. a) $\left.\begin{array}{l}x + y = 120°\\ \sin x - \sin y = \dfrac{1}{2}\end{array}\right\} \Rightarrow \left.\begin{array}{l}x + y = 120°\\ 2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2} = \dfrac{1}{2}\end{array}\right\} \Rightarrow$

$$\Rightarrow 2\cos 60°\sin\frac{x-y}{2} = \frac{1}{2} \Rightarrow 2\cdot\frac{1}{2}\cdot\sin\frac{x-y}{2} = \frac{1}{2}$$

$$\Rightarrow \sin\frac{x-y}{2} = \frac{1}{2} \Rightarrow \frac{x-y}{2} = \text{arc sin}\left(\frac{1}{2}\right) \Rightarrow x - y = 60°$$

$$\Rightarrow \left.\begin{array}{l}x + y = 120°\\ x - y = 60°\end{array}\right\} \Rightarrow x = 90°,\ y = 30°$$

Expressem els resultats en radians:

$$x = 90\cancel{°}\cdot\frac{2\pi\text{ rad}}{360\cancel{°}} = \frac{\pi}{2}\text{ rad},\ y = 30\cancel{°}\cdot\frac{2\pi\text{ rad}}{360\cancel{°}} = \frac{\pi}{6}\text{ rad}$$

b) $\left.\begin{array}{l}\cos^2 x + \sin^2 y = 1\\ \cos^2 x - \sin^2 y = 0\end{array}\right\}$

Aquest sistema té infinites solucions. Si restem les dues equacions, veiem que s'obté una equació del tipus $\cos^2 x + \sin^2 x = 1$, i això és cert sempre per a qualsevol valor de x. Per tant, qualsevol parella de x i y que siguin complementaris compleix aquest sistema.

66. $\text{cotg}\ x - 2\,\text{cosec}\ 2x = 1 \rightarrow \dfrac{\cos x}{\sin x} - \dfrac{2}{\sin 2x} = 1$

$$\frac{\cos x}{\sin x} - \frac{1}{\sin x\cdot\cos x} = 1 \rightarrow \frac{\cos^2 x - 1}{\sin x\cdot\cos x} = \frac{\sin x\cdot\cos x}{\sin x\cdot\cos x}$$

$$-\sin^2 x = \sin x\cdot\cos x \rightarrow \sin x(\cos x + \sin x) = 0$$

Descartem la solució $\sin x = 0$, ja que, si la substituïm en l'equació inicial, no està definida.

$$\cos x + \sin x = 0 \rightarrow \text{tg}\ x = -1 \rightarrow x = \text{arc tg}\,(-1)$$

$$x = \frac{3\pi}{4} + k\pi$$

67. a) $\left.\begin{array}{l}\sin x + \sin y = \sqrt{3}\\ \cos x + \cos y = 1\end{array}\right\} \Rightarrow \left.\begin{array}{l}2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2} = \sqrt{3}\\ 2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2} = 1\end{array}\right\} \Rightarrow$

$$\Rightarrow \frac{\cancel{2}\sin\dfrac{x+y}{2}\cancel{\cos\dfrac{x-y}{2}}}{\cancel{2}\cos\dfrac{x+y}{2}\cancel{\cos\dfrac{x-y}{2}}} = \frac{\sqrt{3}}{1} \Rightarrow \text{tg}\,\frac{x+y}{2} = \sqrt{3}$$

$$\Rightarrow \frac{x+y}{2} = \text{arc tg}\left(\sqrt{3}\right) \Rightarrow x + y = \frac{2\pi}{3}$$

$$\Rightarrow \left.\begin{array}{l}2\sin\dfrac{\pi}{3}\cos\dfrac{x-y}{2} = \sqrt{3}\\ 2\cos\dfrac{\pi}{3}\cos\dfrac{x-y}{2} = 1\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}2\cdot\dfrac{\sqrt{3}}{2}\cdot\cos\dfrac{x-y}{2} = \sqrt{3}\\ 2\cdot\dfrac{1}{2}\cdot\cos\dfrac{x-y}{2} = 1\end{array}\right\} \Rightarrow \left.\begin{array}{l}\cos\dfrac{x-y}{2} = 1\\ \cos\dfrac{x-y}{2} = 1\end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{x-y}{2} = 0 \Rightarrow x = y \Rightarrow \left.\begin{array}{l}x + y = \dfrac{2\pi}{3}\\ x = y\end{array}\right\} \Rightarrow$$

$$\Rightarrow x = \frac{\pi}{3},\ y = \frac{\pi}{3}$$

b) $\left.\begin{array}{l}\sin^2 x + \cos^2 y = \dfrac{3}{4}\\ \cos^2 x - \sin^2 y = \dfrac{1}{4}\end{array}\right\} \Rightarrow$

$$\Rightarrow \sin^2 x - \cos^2 x + \cos^2 y + \sin^2 y = \frac{1}{2}$$

$$\Rightarrow \cos^2 x - \sin^2 x = \frac{1}{2} \Rightarrow \cos 2x = \frac{1}{2} \Rightarrow 2x = \text{arc cos}\left(\frac{1}{2}\right)$$

$$\Rightarrow x = \frac{\pi}{6} \Rightarrow \sin^2\frac{\pi}{6} + \cos^2 y = \frac{3}{4} \Rightarrow \cos^2 y = \frac{1}{2} \Rightarrow y = \frac{\pi}{4}$$

c) $\left.\begin{array}{l}\cos(x + y) = \dfrac{1}{2}\\ \sin(x - y) = \dfrac{1}{2}\end{array}\right\} \Rightarrow \left.\begin{array}{l}x + y = \text{arc cos}\left(\dfrac{1}{2}\right)\\ x - y = \text{arc sin}\left(\dfrac{1}{2}\right)\end{array}\right\} \Rightarrow$

$$\Rightarrow \left.\begin{array}{l}x + y = 60°\\ x - y = 30°\end{array}\right\} \Rightarrow x = 45°,\ y = 15°$$

Expressem els resultats en radians:

$$x = 45\cancel{°}\cdot\frac{2\pi\text{ rad}}{360\cancel{°}} = \frac{\pi}{4}\text{rad},\ y = 15\cancel{°}\cdot\frac{2\pi\text{ rad}}{360\cancel{°}} = \frac{\pi}{12}\text{rad}$$

SÍNTESI Pàg. 120

68. $\cos 4x = \cos 2 \cdot (2x) =$

$= \cos^2(2x) - \sin^2(2x) = \left(\cos^2 x - \sin^2 x\right)^2 - \left(2\sin x \cos x\right)^2 =$

$= \cos^4 x + \sin^4 x - 2\sin^2 x \cos^2 x - 4\sin^2 x \cos^2 x =$

$= \cos^4 x + \sin^4 x - 6\sin^2 x \cos^2 x$

Fem la prova, per exemple, amb 45°.

$$-1 = \cos 4 \cdot 45° = \left(\frac{1}{\sqrt{2}}\right)^4 + \left(\frac{1}{\sqrt{2}}\right)^4 - 6\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)^2 = -1$$

69. Com que els tres angles pertanyen a un triangle, es compleix que $\alpha + \beta + \gamma = 180° \Rightarrow \alpha = 180° - \beta - \gamma$

a) $\sin(\alpha + \beta) - \sin\gamma = \sin(180° - \beta - \gamma + \beta) - \sin\gamma =$

$= \sin(180° - \gamma) - \sin\gamma = \sin\gamma - \sin\gamma = 0$

b) $\cos(\alpha + \beta) + \cos\gamma = \cos(180° - \beta - \gamma + \beta) + \cos\gamma =$

$= \cos(180° - \gamma) + \cos\gamma = -\cos\gamma + \cos\gamma = 0$

c) $\text{tg}(\alpha + \beta) + \text{tg}\,\gamma = \text{tg}(180° - \beta - \gamma + \beta) + \text{tg}\,\gamma =$

$= \text{tg}(180° - \gamma) + \text{tg}\,\gamma = -\text{tg}\,\gamma + \text{tg}\,\gamma = 0$

70. $\dfrac{\sin x + \cos x}{\cos x - \sin x} \cdot \cos 2x - \sin 2x =$

$= \dfrac{\sin x + \cos x}{\cos x - \sin x} \cdot \left(\cos^2 x - \sin^2 x\right) - 2\sin x \cos x =$

$= \dfrac{(\sin x + \cos x)(\sin x + \cos x)\cancel{(\sin x - \cos x)}}{\cancel{\cos x - \sin x}} -$

$- 2\sin x \cos x =$

$= \sin^2 x + \cos^2 x + \cancel{2\sin x \cos x} - \cancel{2\sin x \cos x} = 1$

71. Sabem que $\alpha + \beta + \gamma = 180° \rightarrow \alpha = 180° - (\beta + \gamma)$, aleshores:

$$\text{tg}\,\alpha + \text{tg}\,\beta + \text{tg}\,\gamma = \text{tg}(180° - (\beta + \gamma)) + \text{tg}\,\beta + \text{tg}\,\gamma =$$

$$= -\text{tg}(\beta + \gamma) + \text{tg}\,\beta + \text{tg}\,\gamma = -\text{tg}(\beta + \gamma) + \text{tg}(\beta + \gamma) \cdot (1 - \text{tg}\,\beta \cdot \text{tg}\,\gamma)$$

$$= -\text{tg}(\beta + \gamma) + \text{tg}(\beta + \gamma) - \text{tg}(\beta + \gamma)\,\text{tg}\,\beta\,\text{tg}\,\gamma =$$

$$= -\text{tg}(\beta + \gamma) \cdot \text{tg}\,\beta \cdot \text{tg}\,\gamma = -\text{tg}(180° - \alpha) \cdot \text{tg}\,\beta \cdot \text{tg}\,\gamma = \text{tg}\,\alpha \cdot \text{tg}\,\beta \cdot \text{tg}\,\gamma$$

72. a) $\dfrac{1 + \cos 2x}{\sin 2x} = \dfrac{1 + \cos^2 x - \sin^2 x}{2\sin x \cos x} = \dfrac{\cos^2 x + \cos^2 x}{2\sin x \cos x} =$

$= \dfrac{\cancel{2}\cos \cancel{^2}x}{\cancel{2}\sin x \cancel{\cos x}} = \dfrac{\cos x}{\sin x} = \text{cotg}\,x$

b) $\dfrac{\sin(\pi + \alpha) \cdot \cos\left(\frac{\pi}{2} - \alpha\right)}{\left(\cos^2\alpha - 1\right) \cdot \text{tg}(\pi - \alpha) \cdot \text{cotg}(2\pi - \alpha)} =$

$= \dfrac{-\cancel{\sin\alpha} \cdot \cancel{\sin\alpha}}{-\cancel{\sin^2\alpha} \cdot \left(-\cancel{\text{tg}\,\alpha}\right) \cdot \left(-\frac{1}{\cancel{\text{tg}\,\alpha}}\right)} = \dfrac{-1}{-1} = 1$

c) $\dfrac{\text{tg}(180° - \alpha) \cdot \text{cotg}(360° - \alpha)}{\sec\alpha \cdot \cos(180° - \alpha)} = \dfrac{-\cancel{\text{tg}\,\alpha} \cdot \frac{1}{-\cancel{\text{tg}\,\alpha}}}{\frac{1}{\cancel{\cos\alpha}} \cdot \left(-\cancel{\cos\alpha}\right)} =$

$= 1/-1 = -1$

73. $\sin 4\alpha = \sin(2\alpha + 2\alpha) = \sin 2\alpha \cos 2\alpha + \cos 2\alpha \sin 2\alpha =$

$= 2\sin\alpha\cos\alpha \cdot \left(\cos^2\alpha - \sin^2\alpha\right) + \left(\cos^2\alpha - \sin^2\alpha\right) \cdot$

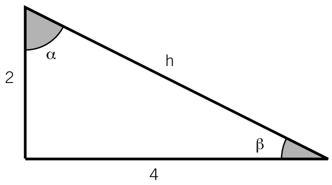
$\cdot\, 2\sin\alpha\cos\alpha = 4\sin\alpha\cos\alpha \cdot \left(\cos^2\alpha - \sin^2\alpha\right) =$

$= 4\sin\alpha\cos^3\alpha - 4\sin^3\alpha\cos\alpha$

74. $\dfrac{\cos 75° + \cos 15°}{\cos 75° - \cos 15°} = \dfrac{2\cos\left(\frac{75° + 15°}{2}\right) \cdot \cos\left(\frac{75° - 15°}{2}\right)}{-2\sin\left(\frac{75° + 15°}{2}\right) \cdot \sin\left(\frac{75° - 15°}{2}\right)} =$

$= \dfrac{2\cos 45° \cos 30°}{-2\sin 45° \sin 30°} = \dfrac{2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}}{-2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{1}{2}} = -\sqrt{3}$

75. En primer lloc, calculem el valor de la hipotenusa per poder trobar posteriorment les raons trigonomètriques dels angles aguts del triangle. Per a aquests, hi apliquem el teorema de Pitàgores: $h = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$



$\sin\alpha = \dfrac{\text{c. oposat}}{\text{hipotenusa}} = \dfrac{4}{2\sqrt{5}} = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$

$\cos\alpha = \dfrac{\text{c. contigu}}{\text{hipotenusa}} = \dfrac{2}{2\sqrt{5}} = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$

$\text{tg}\,\alpha = \dfrac{\text{c. oposat}}{\text{c. contigu}} = \dfrac{4}{2} = 2$

$\sin\beta = \dfrac{\text{c. oposat}}{\text{hipotenusa}} = \dfrac{2}{2\sqrt{5}} = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$

$\cos\beta = \dfrac{\text{c. contigu}}{\text{hipotenusa}} = \dfrac{4}{2\sqrt{5}} = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$

$\text{tg}\,\beta = \dfrac{\text{c. oposat}}{\text{c. contigu}} = \dfrac{2}{4} = \dfrac{1}{2}$

76. L'angle α es troba en el segon quadrant; per tant, el valor del sinus serà positiu i el valor del cosinus, negatiu.

$$\left.\begin{array}{l}\text{tg}\,\alpha = \dfrac{\sin\alpha}{\cos\alpha} = -2\\ \sin^2\alpha + \cos^2\alpha = 1\end{array}\right\} \Rightarrow \left.\begin{array}{l}\cos\alpha = -\dfrac{\sin\alpha}{2}\\ \sin^2\alpha + \cos^2\alpha = 1\end{array}\right\}$$

$$\Rightarrow \sin^2\alpha + \left(-\frac{\sin\alpha}{2}\right)^2 = 1 \Rightarrow 5\sin^2\alpha = 4 \Rightarrow$$

$$\Rightarrow \sin\alpha = \sqrt{\frac{4}{5}} = 0{,}89 \Rightarrow \cos\alpha = -\frac{0{,}89}{2} = -0{,}45$$

$$\sin(180° - \alpha) = \sin\alpha = 0{,}89;\ \cos(180° - \alpha) = -\cos\alpha = 0{,}45$$

$$\text{tg}(180° - \alpha) = -\text{tg}\,\alpha = 2$$

$$\sin(360° - \alpha) = -\sin\alpha = -0{,}89;\ \cos(360° - \alpha) = \cos\alpha = -0{,}45$$

$$\text{tg}(360° - \alpha) = -\text{tg}\,\alpha = 2$$

Avaluació (pàg. 122)

1. a) $30° = 30\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{\pi}{6} \text{rad}$

b) $72° = 72\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{2\pi}{5} \text{rad}$

c) $127° = 127\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{127\pi}{180} \text{rad}$

d) $200° = 200\not{°} \cdot \frac{2\pi \text{ rad}}{360\not{°}} = \frac{10\pi}{9} \text{rad}$

2. a) $3 \text{ rad} = 3\not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 171{,}887°$

b) $\frac{\pi}{13} \text{ rad} = \frac{\pi}{13} \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 13{,}846°$

c) $0{,}36 \text{ rad} = 0{,}36\not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 20{,}626°$

d) $\frac{5\pi}{7} \text{ rad} = \frac{5\pi}{7} \not{\text{rad}} \cdot \frac{360°}{2\pi \not{\text{rad}}} = 128{,}571°$

3. Els angles de 12° i 37° se situen en el primer quadrant; per tant, el cosinus i la tangent seran positius.

a) Hi apliquem el teorema fonamental de la trigonometria:

$$\sin^2 12° + \cos^2 12° = 1 \Rightarrow (0{,}2)^2 + \cos^2 12° = 1$$
$$\Rightarrow \cos 12° = \sqrt{1-(0{,}2)^2} = 0{,}98$$

b) $\text{tg}\, 12° = \frac{\sin 12°}{\cos 12°} = \frac{0{,}2}{0{,}98} = 0{,}204$

c) Hi apliquem el teorema fonamental de la trigonometria:

$$\sin^2 37° + \cos^2 37° = 1 \Rightarrow (0{,}6)^2 + \cos^2 37° = 1$$
$$\Rightarrow \cos 37° = \sqrt{1-(0{,}6)^2} = 0{,}8$$

4. Per a aquest exercici, utilitzem el teorema fonamental de la trigonometria, sabent que l'angle de 72° pertany al primer quadrant i, per tant, el sinus i la tangent seran positius:

$$\sin^2 78° + \cos^2 78° = 1 \Rightarrow \sin^2 78° + (0{,}2)^2 = 1$$
$$\Rightarrow \sin 78° = \sqrt{1-(0{,}2)^2} = 0{,}98$$
$$\Rightarrow \text{tg}\, 78° = \frac{\sin 78°}{\cos 78°} = \frac{0{,}98}{0{,}2} = 4{,}9$$

5. a) A partir de la taula de la pàgina 105 d'aquest tema, podem efectuar l'operació següent:

$$5\cos\frac{\pi}{2} + 2\cos\pi - \cos 0 - \cos\frac{3\pi}{2} + \cos 2\pi =$$
$$= 5 \cdot 0 + 2 \cdot (-1) - 1 - 0 + 1 = -2$$

b) Calculem les raons trigonomètriques que no sabem:

$$\sin\frac{2\pi}{3} = \sin\left(2\cdot\frac{\pi}{3}\right) = 2\cdot\sin\left(\frac{\pi}{3}\right)\cdot\cos\left(\frac{\pi}{3}\right) =$$
$$= 2\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}$$

Ara, per mitjà de les taules de les pàgines 103 i 105 d'aquest tema, podem efectuar l'operació:

$$2\sqrt{3}\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{2} + 4\sin\frac{\pi}{6} =$$
$$= 2\cdot\sqrt{3}\cdot\frac{\sqrt{3}}{2} - 2\cdot 1 + 4\cdot\frac{1}{2} =$$
$$= 3 - 2 + 2 = 3$$

6. a) cos (100°) = cos (180° – 80°) = –cos 80°

b) cos 930° = cos 210° = cos (180° +30°) = –cos 30°

c) sin 125° = sin (180° – 55°) = sin 55°

d) sin (–280°) = –sin 280° = –sin (360° – 80°) = sin 80°

7. $\frac{\cos(a+b)+\cos(a-b)}{\sin(a+b)+\sin(a-b)} = \frac{\not{2}\cos a \not{\cos b}}{\not{2}\sin a \not{\cos b}} = \frac{\cos a}{\sin a} =$

$= \text{cotg}\, a = \frac{1}{\text{tg}\, a}$

8. Com hem calculat en l'exercici 4 d'aquesta pàgina, tenim que sin 78° = 0,98 i que tg 78° = 4,9. També, de l'enunciat, cos 78° = 0,2.

Calculem les raons trigonomètriques de l'angle meitat, 39°, en què aquest angle pertany al primer quadrant i, per tant, tots els valors de les raons trigonomètriques seran positius.

$$\sin 39° = \sin\frac{78°}{2} = \sqrt{\frac{1-\cos 78°}{2}} = 0{,}632$$
$$\cos 39° = \cos\frac{78°}{2} = \sqrt{\frac{1+\cos 78°}{2}} = 0{,}775$$
$$\text{tg}\, 39° = \text{tg}\frac{78°}{2} = \sqrt{\frac{1-\cos 78°}{1+\cos 78°}} = 0{,}816$$

Calculem les raons trigonomètriques de l'angle doble, 156°, on aquest angle pertany al segon quadrant i, per tant, el sinus serà positiu, mentre que el cosinus i la tangent seran negatius.

$$\sin 156° = \sin(2\cdot 78°) = 2\cdot\sin 78°\cdot\cos 78° = 0{,}392$$
$$\cos 156° = \cos(2\cdot 78°) = \cos^2 78° - \sin^2 78° = -0{,}92$$
$$\text{tg}\, 156° = \text{tg}(2\cdot 78°) = \frac{2\,\text{tg}\, 78°}{1-\text{tg}^2\, 78°} = -0{,}426$$

9. a) $\cos^2\left(\frac{x}{2}\right) + \cos x = \frac{1}{2} \Rightarrow \left(\sqrt{\frac{1+\cos x}{2}}\right)^2 + \cos x = \frac{1}{2}$

$\frac{1+\cos x}{2} + \cos x = \frac{1}{2} \Rightarrow 1 + \cos x + 2\cos x = 1 \Rightarrow$

$\Rightarrow \cos x = 0 \Rightarrow x = \text{arc}\cos(0) \Rightarrow x = 90° + 180°\cdot k$

b) $\sin 2x \cdot \cos x = 6\sin^3 x \Rightarrow 2\sin x\cos x\cos x = 6\sin^3 x$

$\sin x\left(\cos^2 x - 3\sin^2 x\right) = 0$

$\sin x = 0 \rightarrow x_1 = \text{arc sin}(0) = k \cdot 180°$

$\cos^2 x - 3\sin^2 x = 0 \Rightarrow 1 - \sin^2 x - 3\sin^2 x = 0 \Rightarrow$

$\Rightarrow 1 = 4\sin^2 x \Rightarrow \sin^2 x = \frac{1}{4} \Rightarrow$

$$\Rightarrow \begin{cases} x = \text{arc sin}\left(\frac{1}{2}\right) \Rightarrow x_2 = 30° + 180° \cdot k \\ x = \text{arc sin}\left(-\frac{1}{2}\right) \Rightarrow x_3 = 150° + 180° \cdot k \end{cases}$$

c) $\text{tg}\left(\frac{\pi}{4} - x\right) + \text{tg}\, x = 1 \Rightarrow \dfrac{\text{tg}\frac{\pi}{4} - \text{tg}\, x}{1 + \text{tg}\frac{\pi}{4}\,\text{tg}\, x} + \text{tg}\, x = 1$

$$\Rightarrow \frac{1 - \text{tg}\, x}{1 + \text{tg}\, x} + \text{tg}\, x = 1 \Rightarrow (1 - \text{tg}\, x)\left(\frac{\text{tg}\, x}{1 + \text{tg}\, x}\right) = 0$$

$$\begin{cases} \text{tg}\, x = 1 \Rightarrow x_1 = \text{arc tg}(1) = 45° + 180° \cdot k \\ \text{tg}\, x = 0 \Rightarrow x_2 = \text{arc tg}(0) = 180° \cdot k \end{cases}$$

d) $\dfrac{\sin 3x + \sin x}{\cos 3x - \cos x} = \sqrt{3} \Rightarrow$

$$\Rightarrow \frac{2\sin\frac{3x + x}{2} \cdot \cos\frac{3x - x}{2}}{-2\sin\frac{3x + x}{2} \cdot \sin\frac{3x - x}{2}} = \sqrt{3}$$

$$\Rightarrow -\frac{\cos x}{\sin x} = \sqrt{3} \Rightarrow -\text{cotg}\, x = \sqrt{3} \Rightarrow -\frac{1}{\text{tg}\, x} = \sqrt{3}$$

$$\Rightarrow \text{tg}\, x = -\frac{1}{\sqrt{3}} \Rightarrow x = 150° + 180° \cdot k$$

10. $\sin^2 x = 1 - \cos x \Rightarrow 1 - \cos^2 x = 1 - \cos x \Rightarrow \cos^2 x = \cos x$

$\Rightarrow \cos^2 x - \cos x = 0 \Rightarrow \cos x(\cos x - 1) = 0$

$$\Rightarrow \begin{cases} \cos x = 1 \Rightarrow x_1 = 0,\ x_2 = 2\pi \\ \cos x = 0 \Rightarrow x_3 = \frac{\pi}{2},\ x_4 = \frac{3\pi}{2} \end{cases}$$

11. a) $\left.\begin{aligned} \sin^2 x + \cos^2 y = \frac{5}{4} \\ \sin^2 x - \cos^2 y = \frac{3}{4} \end{aligned}\right\} \Rightarrow 2\sin^2 x = 2 \Rightarrow \sin^2 x = 1$

$\Rightarrow \sin x = \pm 1 \Rightarrow x = \text{arc sin}(\pm 1) \Rightarrow x = 90° + 180° \cdot k$

$$\Rightarrow \cos^2 y = \frac{1}{4} \Rightarrow \left.\begin{aligned} y = \text{arc cos}\left(\frac{1}{2}\right) \\ y = \text{arc cos}\left(-\frac{1}{2}\right) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} y = 60° + 180° \cdot k \\ y = 120° + 180° \cdot k \end{aligned}\right\}$$

b) $\left.\begin{aligned} \text{tg}\, x + \text{tg}\, y = 2 \\ x - y = \pi \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \text{tg}\, x + \text{tg}\, y = 2 \\ x - \pi = y \end{aligned}\right\} \Rightarrow \text{tg}\, x + \text{tg}(x - \pi) = 2$

$\Rightarrow \text{tg}\, x + \text{tg}\, x = 2 \Rightarrow 2\,\text{tg}\, x = 2 \Rightarrow \text{tg}\, x = 1 \Rightarrow$

$\Rightarrow x = \text{arc tg}(1) \Rightarrow x = 225° + 180° \cdot k \Rightarrow$

$\Rightarrow y = 225° + 180° \cdot k - 180° = 45° + 180° \cdot k$

12. a) $\dfrac{\sin^2(\pi - \alpha) \cdot \cos\left(\frac{\pi}{2} - \alpha\right) \cdot \text{tg}(\pi + \alpha)}{\sin\alpha \cdot (1 - \cos^2\alpha) \cdot \sin(2\pi - \alpha)} =$

$$= \frac{\sin^2\alpha \cdot \sin\alpha \cdot \text{tg}\,\alpha}{\sin\alpha \cdot \sin^2\alpha \cdot (-\sin\alpha)} = -\frac{\text{tg}\,\alpha}{\sin\alpha} = -\frac{1}{\cos\alpha} = -\sec\alpha$$

b) $\text{tg}\left(\frac{\pi}{2} - \alpha\right) \cdot \text{tg}(\pi - \alpha) - \dfrac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\cos(\pi + \alpha)} =$

$$= \frac{1}{\text{tg}\,\alpha} \cdot (-\text{tg}\,\alpha) - \frac{\cos\alpha}{-\cos\alpha} = -1 - (-1) = 0$$

Zona + (pàg. 123)

— **La trigonometria i la cardiologia**

- La relació trigonomètrica és la següent: $\cos^{-1}(r^2/R^2)$, en què r és el radi de l'artèria més petita i R és el radi de l'artèria més gran, si bé aquesta relació únicament serà vàlida si $r < R$.

  La fórmula només és útil si el valor del cosinus de l'angle està entre −1 i 1, ja que el cosinus únicament està definit entre aquests dos valors.

— **Distàncies estel·lars**

- La fórmula trigonomètrica que intervé en la mesura de distàncies estel·lars és la següent:

  Dist. estel (*UA*) = (dist. Terra-Sol)/tg(*p*)

  En què *p* = paral·laxi en segons d'arc i *UA* = unitat astronòmica (dist. Terra-Sol)

- Calculem la distància des de la Terra fins a l'estel Èpsilon Eridani:

  Dist. estel = 1 UA/tg (0° 0' 0,31") = 665 370 UA

  Com que un any llum = 63 271 UA, aleshores:

  665 370 UA/63 271 UA = 10,52 anys llum

  Com que un parsec = 3,259 anys-llum, aleshores:

  10,52 anys llum/3,259 anys llum = 3,23 parsecs